ISRAEL JOURNAL OF MATHEMATICS 140 (2004), 157–202

COMPOSITION SERIES OF GENERALIZED PRINCIPAL SERIES; THE CASE OF STRONGLY POSITIVE DISCRETE SERIES

BY

GORAN MUIĆ

Department of Mathematics, University of Zagreb
Bijenicka bb, 10000 Zagreb, Croatia
e-mail: gmuic@bete.math.hr

ABSTRACT

In this paper we determine the composition series of the generalized principal series $\delta \rtimes \sigma$ assuming that σ is strongly positive discrete series.

Introduction

This paper is the first in the series of papers in which we determine reducibilities of non-unitary generalized principal series for classical p-adic groups in terms of the recent classification of discrete series due to Mœglin and Tadić ([Mœ], [MT]).

To describe our results, we introduce some notation. Let G_n be a symplectic or (full) orthogonal group having split rank n. Let $\sigma \in \operatorname{Irr} G_n$ be a discrete series. We write $(\mathrm{Jord}, \sigma', \epsilon)$ for its admissible triple associated to σ by Mœglin (see [Mœ] or Section 1 here). Let $\delta \in \operatorname{Irr} GL(m_\delta, F)$ (this defines m_δ) be an essentially square integrable representation. According to [Ze], δ is attached to the segment. We may (and will) write this segment as follows: $[\nu^{-l_1}\rho,\ \nu^{l_2}\rho]$, $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$, $\rho \in \operatorname{Irr} GL(m_\rho, F)$ (this defines m_ρ) is unitary. Since $\delta \rtimes \sigma$ has the same composition series as $\widetilde{\delta} \rtimes \sigma$ we may assume $l_2 - l_1 > 0$. The case of unitary generalized principal series $l_1 - l_2 = 0$ is discussed in [MT].

In this paper we determine in a simple and in a uniform fashion reducibility and composition series of $\delta \rtimes \sigma$, when σ is a strongly positive discrete series (see Section 1). This generalizes and simplifies previous works of Jantzen [J], Tadić

Received April 7, 2003

[T] and others (see [T] for reference). The general case is more complicated and it will be discussed in separate papers (see [M4] and [M5]). This paper is an important step towards the general case as, in the approach to discrete series adapted in ([Mœ], [MT]), strongly positive discrete series are basic building blocks of inductive construction of discrete series (see Theorem 1.1). They, for example, include representations such as supercuspidal, discrete series obtained by Howe correspondence from supercuspidal representations (see for example [M3]), regular discrete series, generalized Steinberg representations etc.

The structure of the composition series $\delta \rtimes \sigma$ depends on Jord_ρ, where as usual (see [MT]) this stands for the set of all positive half-integers $2a+1$ such that $(2a+1, \rho) \in \mathrm{Jord}$ (see Section 1 for a precise definition of all terms involved). It depends also on the parity condition that can be expressed as requiring that $l_1 - a$ (or $l_2 - a$) is an integer for some (all) $2a+1 \in \mathrm{Jord}_\rho$. We denote (see (2.3))

$$(NT) \qquad \mathrm{Jord}_\rho \neq \emptyset \quad \text{and} \quad l_1 - a \in \mathbb{Z}, \quad \forall 2a+1 \in \mathrm{Jord}_\rho .$$

(NT) can fail for several reasons and all of them are easy to handle in general (that is, not assuming that σ is strongly positive). This is done in Section 2. One particular case when (NT) fails is when ρ is not self-dual or $2l_1 + 1$ is not integral. Then $\delta \rtimes \sigma$ is irreducible. That is proved in Theorem 2.2 but it is actually a well-known old result of Tadić (see [T] and reference there).

So assume now that ρ is self-dual and $2l_1 + 1$ is integral. Then if $\mathrm{Jord}_\rho \neq \emptyset$ and the second condition in (NT) fails, then $\delta \rtimes \sigma$ is irreducible. If $\mathrm{Jord}_\rho = \emptyset$, then $\delta \rtimes \sigma$ is reducible if and only if $2l_1 + 1 \geq 0$ and

$$2l_1 + 1 \in 2\mathbb{Z} \iff L(0, \rho, r) = \infty.$$

The local L-function given here is either an exterior-square or symmetric-square L-function defined by Shahidi [Sh1] (see Section 1 for the precise definition).

In that case the composition series of $\delta \rtimes \sigma$ consists of at most two non-isomorphic discrete series subrepresentations and the Langlands quotient (see Theorem 2.3).

We assume now that (NT) holds. In this paper we determine the composition series of $\delta \rtimes \sigma$ assuming that σ is a strongly positive discrete series. In the third section we consider the case $l_1 \leq -1$, in the fourth section we consider $l_1 \geq 0$, and in the fifth we consider the case $l_1 = -1/2$. The main results with a precise description of irreducible subquotients are stated at the beginning of each section. (See Proposition 3.1, Theorem 4.1, and Theorem 5.1.) We shall give here a quantitative version of the result.

We describe the case $l_1 \leq -1$ first. Then the composition series of $\delta \rtimes \sigma$ depends on $[-2l_1 - 1, 2l_2 + 1] \cap \mathrm{Jord}_\rho$. The induced representation $\delta \rtimes \sigma$ is irreducible if and only if this intersection is empty or equal to $\{2l + 1\}$. Otherwise, it is of length two and multiplicty one. Except for its Langlands quotient it has one more representation that is in a discrete series if and only if $-2l_1 - 1 \in \mathrm{Jord}_\rho$, otherwise it is non-tempered.

Next, we discuss the case $l_1 = -1/2$ and $\epsilon(\min \mathrm{Jord}_\rho, \rho) = -1$. Then $\delta \rtimes \sigma$ is irreducible if and only if $2l + 1 \in \mathrm{Jord}_\rho$. Otherwise, the composition series of $\delta \rtimes \sigma$ consists of its Langlands quotient and an irreducible representation that is in a discrete series if and only if $2l+1 < \min \mathrm{Jord}_\rho$, otherwise it is non-tempered.

The cases $l_1 \geq 0$, and $l_1 = -1/2$ and $\epsilon(\min \mathrm{Jord}_\rho, \rho) = 1$ are more complicated. The composition series of $\delta \rtimes \sigma$ depends on $[2l_1 + 1, 2l_2 + 1] \cap \mathrm{Jord}_\rho$. If this intersection is empty, $\delta \rtimes \sigma$ has two (resp. one) non-isomorphic discrete series subrepresentations if $l_1 \geq 0$ (resp. $l_1 = -1/2$) and its Langlands quotient. If $[2l_1 + 1, 2l_2 + 1] \cap \mathrm{Jord}_\rho$ is not empty, then we consider several cases.

First, $2l_1 + 1, 2l_2 + 1 \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible. Next, if $2l_1 + 1 \notin \mathrm{Jord}_\rho$ and $2l_2 + 1 \in \mathrm{Jord}_\rho$, then it is of length two and multiplicty one. Except for its Langlands quotient it has one more representation that is tempered (not in a discrete series) if and only if $\mathrm{Jord}_\rho \cap\,]2l_1 + 1,\ 2l_2 + 1[\, = \emptyset$, otherwise it is non-tempered. The case $2l_1 + 1 \in \mathrm{Jord}_\rho$ and $2l_2 + 1 \notin \mathrm{Jord}_\rho$ is analogous. If $2l_1 + 1, 2l_2 + 1 \notin \mathrm{Jord}_\rho$, then if $\mathrm{Jord}_\rho \cap\,]2l_1 + 1,\ 2l_2 + 1[$ is a singleton, then the composition consists of three mutually non-isomorphic (non-tempered) Langlands quotients, and two (resp. one) non-isomorphic discrete series representations if $l_1 \geq 0$ (resp. $l_1 = -1/2$). Otherwise it consists of four mutually non-isomorphic Langlands quotients.

As we can see, the composition series here is always bounded by 5. We should point out that for general σ the composition series of $\delta \rtimes \sigma$ can be arbitrary large ([M4], [M5]).

ACKNOWLEDGEMENT: I would like to thank M. Tadić for drawing my attention to this problem and for many conversations that we had during the preparation of this paper.

1. Preliminaries

Let F be a nonarchimedean field of characteristic different from 2. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively. If $x_1, x_2 \in \mathbb{R}$, we denote by $[x_1,\ x_2]$ (resp.

$]x_1, x_2[$) the set of all $x \in \mathbb{R}$ such that $x_1 \leq x \leq x_2$ (resp. $x_1 < x < x_2$). Similarly, we define $]x_1, x_2]$ and $[x_1, x_2[$.

Now, we shall describe the groups that we consider. We look at the usual towers of orthogonal or symplectic groups $G_n = G(V_n)$ that are groups of isometries of F-spaces $(V_n, (\ , \ ))$, $n \geq 0$, where the form $(\ , \ )$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. We fix a set of standard parabolic subgroups in the usual way.

We now fix just a basic notation of the representation theory of general linear groups and use freely now well-known results of [Ze] through the paper. In particular, we write ν for the character obtained by the composition of the determinant character and (normalized as usual) absolute value of F.

If $\rho \in \operatorname{Irr} GL(m_\rho, F)$ is a supercuspidal representation and $k \in \mathbb{Z}_{\geq 0}$, then we define a segment $[\rho, \nu^k \rho]$ as the set $\{\rho, \nu\rho, \ldots, \nu^k \rho\}$. This segment has a uniquely associated essentially square integrable representation $\delta([\rho, \nu^k \rho])$ given as the unique irreducible subrepresentation of $\nu^k \rho \times \cdots \times \nu\rho \times \rho$.

Now, we shall describe briefly the classification of discrete series in

$$\operatorname{Irr}' = \bigcup_{n \geq 1} \operatorname{Irr} G_n.$$

This has been done in ([Mœ], [MT]) under some assumptions on rank-one reducibilities of the representation induced from the supercuspidal. (See [MT] for the precise statement.) There are no assumptions for the discrete series that are subquotients of representations parabolically induced from generic supercuspidal representations thanks to the work of Shahidi [Sh], and, in particular, if they are subquotients of principal series representations.

We start the discussion of discrete series by recalling ([Mœ]) the definition of two invariants of an attached to a discrete series $\sigma \in \operatorname{Irr}'$. First, a partial supercuspidal support of $\sigma_{cusp} \in \operatorname{Irr}'$ is a supercuspidal representation such that there exists an irreducible representation $\pi \in GL(m_\pi, F)$ (this defines m_π) such that σ is a subrepresentation of the induced representation $\pi \rtimes \sigma_{cusp}$. This property determines $\sigma_{cusp} \in \operatorname{Irr}'$ uniquely.

Next, $\operatorname{Jord}(\sigma)$ is defined as a set of all pairs (a, ρ) ($\rho \cong \widetilde{\rho}$ is a supercuspidal representation of some $GL(m_\rho, F)$, $a > 0$ is an integer) such that (a) and (b) hold:

(a) a is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi ([Sh], [Sh1]), where $r = \wedge^2 \mathbb{C}^{m_\rho}$ is the exterior square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $GL(m_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and

$r = \mathrm{Sym}^2 \mathbb{C}^{m_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $GL(m_\rho, \mathbb{C})$ if G_n is odd-orthogonal.

(b) The induced representation

$$\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$$

is irreducible.

The main point of the classification is that discrete series are in one-to-one correspondence with admissible triples (cf. [Mœ]), and our results are also formulated in terms of admissible triples. So, we start recalling the definition of an admissible triple. This will be given in several steps.

First, we look for a collection Trip of all triples $(\mathrm{Jord}, \sigma', \epsilon)$, where

- $\sigma' \in \mathrm{Irr}'$ is a supercuspidal representation.
- Jord is a finite set (perhaps empty) of pairs (a, ρ) ($\rho \cong \widetilde{\rho}$ is supercuspidal of some $GL(m_\rho, F)$, $a > 0$ is an integer) such that a is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$ (see (a) the above). For example, if $\rho = \chi$ is a quadratic character, then a is odd. We will also recall some notation from [MT]. We write $\mathrm{Jord}_\rho = \{a;\ (a, \rho) \in \mathrm{Jord}\}$, and for $a \in \mathrm{Jord}_\rho$ we write a_- for the largest element of Jord_ρ that is strictly less than a (if it exists).
- ϵ is a function defined on a subset of $\mathrm{Jord} \cup \mathrm{Jord} \times \mathrm{Jord}$ into $\{\pm 1\}$ as follows. First, if $(a, \rho) \in \mathrm{Jord}$, then $\epsilon(a, \rho)$ is not defined if and only if a is odd and $(a', \rho) \in \mathrm{Jord}(\sigma')$ for some positive integer a'. Next, ϵ is defined on a pair $(a, \rho), (a', \rho') \in \mathrm{Jord}$ if and only if $\rho = \rho'$ and $a \neq a'$. This ends the definition of the domain of the definition of ϵ. The following compatibility condition must hold for different $a, a', a'' \in \mathrm{Jord}_\rho$:

(i) If $\epsilon(a, \rho)$ is defined (hence $\epsilon(a', \rho)$ is also defined), then the value of ϵ on (a, ρ) and (a', ρ) is $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$. If $\epsilon(a, \rho)$ is not defined, then the value of ϵ on the pair (a, ρ) and (a', ρ) we shall, after [MT], denote also (formally) by $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$.

(ii) $\epsilon(a, \rho)\epsilon(a'', \rho)^{-1} = (\epsilon(a, \rho)\epsilon(a', \rho)^{-1}) \cdot (\epsilon(a', \rho)\epsilon(a'', \rho)^{-1})$.

(iii) $\epsilon(a, \rho)\epsilon(a', \rho)^{-1} = \epsilon(a', \rho)\epsilon(a, \rho)^{-1}$.

Let $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ and $(a, \rho) \in \mathrm{Jord}$, such that a_- is defined, and

$$\epsilon(a, \rho) \cdot \epsilon(a_-, \rho)^{-1} = 1.$$

Now, it is easy to check the following. If we put $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(a, \rho), (a_-, \rho)\}$, and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup \mathrm{Jord}' \times \mathrm{Jord}'$, then $(\mathrm{Jord}', \sigma', \epsilon') \in \mathrm{Trip}$. We say that the triple $(\mathrm{Jord}', \sigma', \epsilon')$ is subordinated to the triple $(\mathrm{Jord}, \sigma', \epsilon)$.

We say that $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ is an admissible triple of **alternated type** if for any ρ such that $\mathrm{Jord}_\rho \neq \emptyset$ the following holds.

- If $a \in \mathrm{Jord}_\rho$ such that a_- is defined, then

$$\epsilon(a,\rho)\cdot\epsilon(a_-,\rho)^{-1} = -1.$$

- There is an increasing bijection ϕ_ρ: $\mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where

$$\mathrm{Jord}'_\rho(\sigma') = \begin{cases} \mathrm{Jord}_\rho(\sigma') \cup \{0\} & \text{if } a \text{ is even and } \epsilon(\min \mathrm{Jord}_\rho, \rho) = 1; \\ \mathrm{Jord}_\rho(\sigma') & \text{otherwise.} \end{cases}$$

Here $\mathrm{Jord}_\rho(\sigma')$ is a set of all positive integers a such that $(a,\rho) \in \mathrm{Jord}(\sigma')$. We write Trip_{alt} for the set of all triples in Trip that has alternated type.

Remark 1.1: The above definition shows that in an alteranted triple $(\mathrm{Jord}, \sigma', \epsilon)$, ϵ is completely determined by Jord and σ'.

We say that the triple $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ dominates the triple $(\mathrm{Jord}'', \sigma', \epsilon'') \in \mathrm{Trip}$ if we can find a sequence of triples sequence of triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, $1 \le i \le k$, such that

- $(\mathrm{Jord}, \sigma', \epsilon) = (\mathrm{Jord}_1, \sigma', \epsilon_1)$.
- $(\mathrm{Jord}_{i+1}, \sigma', \epsilon_{i+1})$ is subordinated to $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, for each i, $1 \le i \le k-1$.
- $(\mathrm{Jord}'', \sigma', \epsilon'') = (\mathrm{Jord}_k, \sigma', \epsilon_k)$.

Finally, we come to the definition of an admissible triple.

We say that $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ is an **admissible triple** if it dominates some triple of alternated type.

We write Trip_{adm} for the set of all triples in Trip that are admissible. Obviously, we have

$$\mathrm{Trip}_{alt} \subseteq \mathrm{Trip}_{adm} \subseteq \mathrm{Trip}\,.$$

Now, the classification of discrete series ([MT], [Mœ]) can be described as follows.

THEOREM 1.1: *There exists a one-to-one correspondence between the set of all discrete series $\sigma \in \mathrm{Irr}'$ and the set of all triples $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{adm}$ denoted by*

$$\sigma = \sigma_{(\mathrm{Jord}, \sigma', \epsilon)}$$

such that the following holds.

(i) $\mathrm{Jord}(\sigma) = \mathrm{Jord}$ *and* $\sigma_{cusp} = \sigma'$.

(ii) *Let* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{alt}$; *then σ can be described explicitly as follows. For each ρ such that $\mathrm{Jord}_\rho \neq \emptyset$, we write the elements of Jord_ρ in increasing order $a_1^\rho < a_2^\rho < \cdots < a_{k_\rho}^\rho$. Now, σ is a unique irreducible subrepresentation of*

$$\times_\rho \times_{i=1}^{k_\rho} \delta([\nu^{(\phi_\rho(a_i^\rho)+1)/2}\rho,\ \nu^{(a_i^\rho-1)/2}\rho]) \rtimes \sigma'.$$

(iii) *Let* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{adm}$ *and* $(a,\rho) \in \mathrm{Jord}$, *such that* a_- *is defined, and* $\epsilon(a,\rho)\cdot\epsilon(a_-,\rho)^{-1} = 1$. *We put* $\mathrm{Jord}'' = \mathrm{Jord}\setminus\{(a,\rho),(a_-,\rho)\}$, *and consider the restriction* ϵ'' *of* ϵ *to* Jord'. *Then* $(\mathrm{Jord}'', \sigma', \epsilon'') \in \mathrm{Trip}_{adm}$, *and*

$$\sigma \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma_{(\mathrm{Jord}'',\sigma',\epsilon'')}. \tag{1.1}$$

Moreover, the induced representation

$$\delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a_- -1)/2}\rho]) \rtimes \sigma_{(\mathrm{Jord}'',\sigma',\epsilon'')}$$

is a direct sum of two non-equivalent tempered representations $\tau_\pm$, *and there exists the unique* $\tau \in \{\tau_-,\ \tau_+\}$ *such that*

$$\sigma \hookrightarrow \delta([\nu^{(a_- +1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \tau. \tag{1.2}$$

We say that a discrete series is **strongly positive** if its admissible triple is of alternated type ([MT]). In particular, strongly positive discrete series are given by Theorem 1.1 (ii). The strongly positive discrete series can be characterized as follows ([Mœ, Section 1] and [MT, Proposition 7.1]).

PROPOSITION 1.1: *Let* σ *be a discrete series attached to* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{adm}$. *Then* σ *is strongly positive (that is,* $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}_{alt}$*) if and only if for any collection of pairs* (z_i, ρ_i), $1 \le i \le k$, *where* $z_i \in \mathbb{R}$ *and* $\rho_i \in \mathrm{Irr}\, GL(m_{\rho_i}, F)$ *(this defines* m_{ρ_i}*) is a unitary supercuspidal representation, such that*

$$\sigma \hookrightarrow \nu^{z_1}\rho_1 \times \cdots \times \nu^{z_k}\rho_k \rtimes \sigma'$$

implies that $z_i > 0$ *for all* i.

The next proposition will be used several times in the paper. It follows easily from the definition of alternated triple (see also Remark 1.1) and Theorem 1.1.

PROPOSITION 1.2: *Let* σ *be a strongly positive discrete series attached to an alternated triple* $(\mathrm{Jord}, \sigma', \epsilon)$. *Assume that* ρ *is such that* $\mathrm{Jord}_\rho \neq \emptyset$. *Let* $a \in \mathrm{Jord}_\rho$, *and* $b \in \mathbb{Z}_{>0}$ *such that* $a - b \in 2\mathbb{Z}$ *and* $b \notin \mathrm{Jord}_\rho$. *Then there exists a unique strongly positive discrete series* σ'' *such that*

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma'') = \mathrm{Jord}_{\rho'}, & \rho' \not\cong \rho, \\ \mathrm{Jord}_\rho(\sigma'') = \mathrm{Jord}_\rho \setminus \{(a,\rho)\} \cup \{(b,\rho)\}. \end{cases}$$

Our main tool for computing composition series is Tadić's theory of Jacquet modules. We end this section recalling his basic result.

Let $R(G_n)$ be its Grothendieck group of admissible representation of finite length. Put

$$R(G) = \bigoplus_{n \geq 0} R(G_n).$$

We will write $\geq$ or $\leq$ for the natural order on $R(G)$. In more details, $\pi_1 \leq \pi_2$, $\pi_1, \pi_2 \in R(G)$, if and only if $\pi_2 - \pi_1$ is a linear combination of the irreducible representations with positive coefficients.

Similarly, we define

$$R(GL) = \bigoplus_{n \geq 0} R(GL(n, F)).$$

Let $\sigma \in \operatorname{Irr} G_n$. Then for each standard maximal parabolic subgroup P_j, $1 \leq j \leq n$, we can identify $R_{P_j}(\sigma)$ with its semisimplification in $R(GL(j, F)) \otimes R(G_{n-j})$. Thus, we can consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{j=1}^{n} R_{P_j}(\sigma) \in R(GL) \otimes R(G).$$

Now, the basic result of Tadić is the following theorem (see [MT] and reference there).

THEOREM 1.2: *Let $\sigma \in \operatorname{Irr} G_n$. Then we decompose*

$$\mu^*(\sigma) = \sum_{\delta', \sigma_1} \delta' \otimes \sigma_1$$

into irreducible constituents in $R(G)$.

Assume that $l_1, l_2 \in \mathbb{R}$, $l_1 + l_2 + 1 \in \mathbb{Z}_{\geq 0}$, and $\rho \in \operatorname{Irr} GL(m_\rho, F)$ (this defines m_ρ) is a supercuspidal representation. Then we have

$$(1.3)\quad \mu^*(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma) = \sum_{\delta', \sigma_1} \sum_{i=0}^{l_1+l_2+1} \sum_{j=0}^{i} \delta([\nu^{i-l_2}\widetilde{\rho}, \nu^{l_1}\widetilde{\rho}]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta' \otimes \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1.$$

(We skip $\delta[\nu^\alpha\rho, \nu^\beta\rho])$ if $\alpha > \beta$.)

2. Necessary conditions for reducibility

In this section we fix the notation that we use throughout the paper. We assume that σ is a discrete series representation attached to the triple $(\mathrm{Jord}, \sigma', \epsilon)$, and $\delta \in \mathrm{Irr}\, GL(m_\delta, F)$ is an essentially square integrable representation. We study the reducibility and composition series of $\delta \rtimes \sigma$.

By [Ze], δ is attached to the segment. We may (and will) write this segment as follows:

$$(2.1) \qquad [\nu^{-l_1}\rho,\ \nu^{l_2}\rho],\ l_1, l_2 \in \mathbb{R},\ l_1 + l_2 \in \mathbb{Z}_{\geq 0},\ \rho \in \mathrm{Irr}\, GL(m_\rho, F) \text{ unitary}.$$

Next, since $\delta \rtimes \sigma = \tilde{\delta} \rtimes \sigma$ in $R(G)$, we also assume

$$(2.2) \qquad l_2 - l_1 > 0.$$

In this way $\delta \rtimes \sigma$ becomes a standard representation, and we denote by $Lang(\delta \rtimes \sigma)$ its Langlands quotient.

In this section we will reduce the computation reducibility to the (non-trivial) case when the following hold:

$$(2.3) \qquad \begin{cases} \mathrm{Jord}_\rho \neq \emptyset, \\ l_1 - a \in \mathbb{Z}, \quad \forall 2a+1 \in \mathrm{Jord}_\rho . \end{cases}$$

Note that $\mathrm{Jord}_\rho \neq \emptyset$ implies $\rho \cong \tilde{\rho}$, and the second assumption in (2.3) implies $2l_1 + 1, 2l_2 + 1 \in \mathbb{Z}$.

Now, we start the discussion of reducibility of $\delta \rtimes \sigma$ analyzing the form of the possible irreducible subquotients of $\delta \rtimes \sigma$ other than its Langlands quotient. We will look for such a subquotient π requiring that

$$(2.4) \qquad \pi \hookrightarrow \delta_1 \times \pi',$$

where

(a) π' is an irreducible representation.

(b) δ_1 is an essentially square integrable representation attached to a segment $[\nu^{-\alpha_1}\rho_1,\ \nu^{\beta_1}\rho_1]$, $\alpha_1, \beta_1 \in \mathbb{R}$, $\alpha_1 + \beta_1 \in \mathbb{Z}_{\geq 0}$, $-\alpha_1 + \beta_1 < 0$, $\rho_1 \in \mathrm{Irr}\, GL(m_{\rho_1}, F)$ is a unitarizable supercuspidal representation.

We also require that either

$$(2.5) \qquad -\alpha_1 + \beta_1 < 0$$

or

$$(2.6) \qquad \pi' \text{ is tempered and } -\alpha_1 + \beta_1 = 0.$$

This way we try to determine possible tempered (but not square-integrable) or non-tempered subquotients.

Applying Frobenius reciprocity to (2.5), we obtain that $\delta_1 \otimes \pi'$ appears in $\mu^*(\pi)$. Since π is an irreducible subquotient of $\delta \rtimes \sigma$, we obtain that $\delta_1 \otimes \pi'$ is an irreducible subquotient of $\mu^*(\delta \rtimes \sigma)$. Applying Theorem 1.2, we see that there exists an irreducible constituent $\delta' \otimes \sigma_1$ of $\mu^*(\sigma)$, and indices i, j, $0 \leq j \leq i \leq l_1 + l_2 + 1$, such that

$$\begin{cases} \delta_1 \leq \delta([\nu^{i-l_2}\widetilde{\rho}, \nu^{l_1}\widetilde{\rho}]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta', \\ \pi' \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1. \end{cases} \tag{2.7}$$

We now have several cases.

• $i \leq l_1 + l_2$, $j \geq 1$. The formula in (2.7) shows $\rho \cong \rho_1$, $\widetilde{\rho} \cong \rho_1$, and $\alpha_1 - l_2 \in \mathbb{Z}$. Moreover, standard properties of a supercuspidal support of δ_1 shows that $l_1 + 1 \leq l_2 + 1 - j$ (no repetition of the terms there), and $\beta_1 \geq l_2$. Since δ_1 is non-degenerate, we obtain that δ' is non-degenerate. Moreover, the first formula in (2.7) and the classification of non-degenerate representations [Ze] imply

$$\delta' \cong \delta([\nu^{-\alpha_1}\rho, \nu^{i-l_2-1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_2+1}\rho, \nu^{\beta_1}\rho]). \tag{2.8}$$

The first essentially square integrable representation in the product given in (2.8) is non-trivial if and only if $i - l_2 - 1 \geq -\alpha$. If this is so, [Ze] implies that

$$\delta' \hookrightarrow \nu^{i-l_2-1}\rho \times \cdots \times \nu^{-\alpha_1}\rho \times \delta''$$

for some irreducible representation δ''. Since $\delta' \otimes \sigma_1$ is an irreducible constituent of $\mu^*(\sigma)$, we see that

$$\sigma \hookrightarrow \nu^{i-l_2-1}\rho \times \cdots \times \nu^{-\alpha_1}\rho \rtimes \sigma''$$

for some irreducible representation σ''. Now, a square-integrable criterion [Ca] implies $i - l_2 - 1 - \alpha_1 > 0$. On the other hand,

$$i - l_2 - 1 - \alpha_1 \leq (l_1 + l_2) - l_2 - 1 - \alpha_1 = l_1 - 1 - \alpha_1 < \beta_1 - \alpha_1 \leq 0,$$

and this is a contradiction. Thus, $i - l_2 = -\alpha_1$. Hence $i = -\alpha_1 + l_2$.

Since $i \geq 0$ (see Theorem 1.2), we obtain $\alpha_1 \leq l_2$. Next, (2.8) shows $l_2 \leq \beta_1$. Combining this, we get $-\alpha_1 + \beta_1 \geq 0$. This is a contradiction unless we look for a tempered irreducible subquotient. In this case, we obtain $\alpha_1 = \beta_1 = l_2$. Now, $i = 0$, and since $i \geq j \geq 0$, we obtain $j = 0$ too. Now, the second formula of (2.7) implies $\sigma_1 \cong \pi'$. We summarize this discussion as follows. First, ρ is self-dual, and there must exist an irreducible tempered representation σ_1 such that

$$\begin{cases} \pi \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1, \\ \mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1. \end{cases}$$

Note that $l_2 \geq l_1 + 1$ by (2.2). This implies that (2.3) must hold also. Next, we may argue exactly as in ([MT], Section 8), to show that σ_1 is in a discrete series (unless $l_1 \geq 0$ and $2l_1 + 1 \in \mathrm{Jord}_\rho$) satisfying $2l_2 + 1 \in \mathrm{Jord}_\rho$, and $\mathrm{Jord}(\sigma_1)$ is equal to

$$\begin{cases} \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho)\} \cup \{(2l_1+1, \rho)\}, & l_1 \geq 0, \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho)\}, & l_1 = -1/2, \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1, \rho), (-2l_1-1, \rho)\}, & l_1 \leq -1. \end{cases}$$

If $l_1 \geq 0$ and $2l_1 + 1 \in \mathrm{Jord}_\rho$, then σ_1 is a tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1, \rho), (2l_2+1, \rho)\}.$$

• $i = l_1 + l_2 + 1, j \geq 1$. The formula in (2.7) shows $\alpha_1 - l_2 \in \mathbb{Z}$. Moreover, as in the previous case we obtain

$$\delta' \cong \delta([\nu^{-\alpha_1}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_2+1}\rho, \nu^{\beta_1}\rho]).$$

As in the previous case we easily establish $-\alpha_1 = l_2 - j + 1$. (So the first segment does not exist.) Now,

$$\alpha_1 = j - l_2 - 1 \leq (l_1 + l_2 + 1) - l_2 - 1 = l_1 < l_2 \leq \beta_1.$$

Hence $-\alpha_1 + \beta_1 > 0$. This is a contradiction.

• The case $i = l_1 + l_2 + 1, j = 0$ is not possible, since we would obtain $\delta_1 \cong \delta'$, and this implies

$$\sigma \hookrightarrow \nu^{\beta_1}\rho \times \cdots \times \nu^{-\alpha_1}\rho \times \sigma'',$$

for some irreducible representation σ''. This violates the square-integrable criterion [Ca] since $-\alpha_1 + \beta_1 \leq 0$.

• $i \leq l_1 + l_2, j = 0$. The discussion of the present case is similar to the first case considered above. First, using (2.7), we obtain $\widetilde{\rho} \cong \rho_1$, $\beta_1 \geq l_1$, and

$$\delta' \cong \delta([\nu^{-\alpha_1}\widetilde{\rho}, \nu^{i-l_2-1}\widetilde{\rho}]) \times \delta([\nu^{l_1+1}\widetilde{\rho}, \nu^{\beta_1}\widetilde{\rho}]). \tag{2.9}$$

Arguing as in the case $i \leq l_1 + l_2, j \geq 1$, we obtain $\alpha_1 = l_2 - i$. Since $0 \leq i \leq l_1 + l_2$, this is equivalent to

$$-l_1 \leq \alpha_1 \leq l_2. \tag{2.10}$$

Since $\beta_1 \geq l_1$, (2.10) implies $\alpha_1 = \beta_1 = l_1$ if (2.6) holds. Now, assume (2.6) holds. Then, (2.9) shows that δ' is trivial, and thus Theorem 1.2 shows that $\sigma_1 \cong \sigma$. Now, (2.7) shows that there exists a tempered irreducible representation σ_1 such that

$$\begin{cases} \pi \hookrightarrow \delta([\nu^{-l_1}\widetilde{\rho}, \nu^{l_1}\widetilde{\rho}]) \rtimes \sigma_1, \\ \sigma_1 \leq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma. \end{cases}$$

Again using ([MT], Section 8), we easily see that $2l_1 + 1 \in \mathrm{Jord}_\rho$. Also, we see that if $2l_2 + 1 \notin \mathrm{Jord}_\rho$, then σ_1 is in a discrete series satisfying

$$\mathrm{Jord}(\sigma_1) = \mathrm{Jord}(\sigma) \setminus \{(2l_1 + 1, \rho)\} \cup \{(2l_2 + 1, \rho)\},$$

and if $2l_2+1 \in \mathrm{Jord}_\rho$, then σ_1 is a tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1 + 1, \rho), (2l_2 + 1, \rho)\}.$$

Now, assume (2.5) holds, i.e. $-\alpha_1 + \beta_1 < 0$. Then, using (2.10) we obtain $l_1 < \alpha_1$. If $\beta > l_1$, then (2.9) implies that

$$\sigma \hookrightarrow \nu^{\beta_1}\rho \times \cdots \times \nu^{l_1+1}\rho \times \sigma'',$$

for some irreducible representation σ''. Thus, if we have $\beta_1 \geq l_1 + 1$ we obtain $\beta_1 > -(l_1 + 1)$ by square-integrable criterion. Hence, if $\beta_1 \geq l_1 + 1$, we obtain $\beta_1 \geq |l_1 + 1|$ and $2\beta_1 + 1 \in \mathrm{Jord}_\rho$.

Thus, we have established the following results.

LEMMA 2.1: *Assume that π is a tempered (but not square-integrable) irreducible subquotient of $\delta \rtimes \sigma$. Then one of the following must hold.*

(a) $2l_2 + 1 \in \mathrm{Jord}_\rho$ *and* $\pi \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$, *where σ_1 is a tempered representation satisfying*

$$\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \sigma_1. \tag{2.11}$$

Moreover, σ_1 is in a discrete series (unless $l_1 \geq 0$ and $2l_1+1 \in \mathrm{Jord}_\rho$) and $\mathrm{Jord}(\sigma_1)$ *is equal to*

$$\begin{cases} \mathrm{Jord}(\sigma) \setminus \{(2l_2+1,\rho)\} \cup \{(2l_1+1,\rho)\}, & l_1 \geq 0, \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1,\rho)\}, & l_1 = -1/2, \\ \mathrm{Jord}(\sigma) \setminus \{(2l_2+1,\rho),(-2l_1-1,\rho)\}, & l_1 \leq -1. \end{cases}$$

If $l_1 \geq 0$ and $2l_1+1 \in \mathrm{Jord}_\rho$, then σ_1 is a tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho),(2l_2+1,\rho)\}.$$

(b) *$l_1 \geq 0, 2l_1+1 \in \mathrm{Jord}_\rho$, and $\pi \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1$, where σ_1 is a tempered irreducible subquotient of $\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. Moreover, if $2l_2+1 \notin \mathrm{Jord}_\rho$, then σ_1 is in a discrete series satisfying*

$$\mathrm{Jord}(\sigma_1) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho)\} \cup \{(2l_2+1,\rho)\},$$

and if $2l_2+1 \in \mathrm{Jord}_\rho$, then σ_1 is a tempered representation, given as an irreducible subrepresentation of

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_2,$$

where σ_2 is a discrete series satisfying

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma) \setminus \{(2l_1+1,\rho),(2l_2+1,\rho)\}.$$

LEMMA 2.2:

(i) *If π is a non-tempered irreducible subquotient of $\delta \rtimes \sigma$ such that (2.4) and (2.5) hold, then $\rho_1 \cong \rho \cong \widetilde{\rho}$, and there exists an irreducible representation σ_1 such that*

$$\begin{cases} \mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho,\ \nu^{\beta_1}\rho]) \otimes \sigma_1, \\ \pi' \leq \delta([\nu^{\alpha_1+1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma_1. \end{cases} \tag{2.12}$$

Next,

$$\begin{cases} l_1 \leq \beta_1, \\ l_2 \geq \alpha_1 > \beta_1, l_1, \\ \alpha_1 \geq -l_1. \end{cases} \tag{2.13}$$

Moreover, assume $\beta_1 > l_1$. Then (2.3) holds, and $\beta_1 > -l_1 - 1$ and $2\beta_1 + 1 \in \mathrm{Jord}_\rho$. Moreover, if $l_1 < -1/2$, then $2(\beta_1)_- + 1 := (2\beta_1 + 1)_-$ is defined, and we have the following

$$\begin{cases} -l_1 - 1 \le (\beta_1)_- < \beta_1 < \alpha_1 \le l_2, \\ \epsilon(2\beta_1 + 1, \rho)\epsilon(2(\beta_1)_- + 1, \rho)^{-1} = 1. \end{cases}$$

(ii) *Assume that π is a non-tempered irreducible subquotient of $\delta \rtimes \sigma$. We write π as a Langlands subrepresentation of $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \pi_t$, where we have*

- *π_t is a tempered representation.*
- *δ_i, $1 \le i \le k$, is an essentially square integrable representation attached to a segment $[\nu^{-\alpha_i}\rho_i, \ \nu^{\beta_i}\rho_i]$, where $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i + \beta_i \in \mathbb{Z}_{\ge 0}$, $\rho_i \in \mathrm{Irr}\, GL(m_{\rho_i}, F)$ is a unitarizable supercuspidal representation.*
- *$-\alpha_1 + \beta_1 \le -\alpha_2 + \beta_2 \le \cdots \le -\alpha_k + \beta_k < 0$.*

Now, $\rho_1 \cong \rho$. Moreover, taking a unique irreducible (Langlands) subrepresentation

$$\pi' \hookrightarrow \delta_2 \times \cdots \times \delta_k \rtimes \pi_t,$$

we obtain $\pi \hookrightarrow \delta_1 \rtimes \pi'$, and we may apply (i). *Assume further $\beta_1 = l_1$. Then $\rho_2 \cong \rho$ and we have the following.*

If π' is tempered, then $2\alpha_1 + 1 \in \mathrm{Jord}_\rho$ and (2.3) holds.

If π' is not tempered, then $2\beta_2 + 1 \in \mathrm{Jord}_\rho$ and (2.3) holds. Moreover, $2(\beta_2)_- + 1 = (2\beta_2 + 1)_- \in \mathrm{Jord}_\rho$ is defined (see Section 1), and we have the following:

$$\begin{cases} \alpha_1 \le (\beta_2)_- < \beta_2 < \alpha_2 \le l_2, \\ \epsilon(2\beta_2 + 1, \rho)\epsilon(2(\beta_2)_- + 1, \rho)^{-1} = 1. \end{cases}$$

Proof: (i) is a direct consequence of the discussion before Lemma 2.1. (ii) follows from (i). In more detail, since we assume $\beta_1 = l_1$, using (2.12) we obtain that π' is an irreducible subquotient of

$$\delta([\nu^{\alpha_1+1}\rho, \ \nu^{l_2}\rho]) \rtimes \sigma. \tag{2.14}$$

Now, if π' is tempered, we apply results of [MT, Section 8] directly. If π' is not tempered then we may take a unique (Langlands) subrepresentation $\pi'' \hookrightarrow \delta_3 \times \cdots \times \delta_k \rtimes \pi_t$, and we have $\pi' \hookrightarrow \delta_2 \rtimes \pi''$. Now, we again apply (i) considering π' as an irreducible subquotient of the induced representation of (2.14). Of course, we need to adapt the notation properly (see (2.1)). Now, the last inequality in (2.13) reads $\alpha_1 + 1 \le \alpha_2$. Since we assume $-\alpha_1 + \beta_1 \le -\alpha_2 + \beta_2$ and $\alpha_1 + \beta_1 \ge 0$, we have

$$\beta_2 \ge -\alpha_1 + (\beta_1 + \alpha_2) \ge -\alpha_1 + (\beta_1 + \alpha_1 + 1) > -\alpha_1.$$

In particular, $\beta_2 > -\alpha_1 - 1$. Now, we apply (i) directly to complete the proof. ∎

A direct consequence of Lemma 2.2 is the following fact.

COROLLARY 2.1: *In order for $\delta \rtimes \sigma$ to have an irreducible non-tempered subquotient non-isomorphic to $Lang(\delta \rtimes \sigma)$, it is necessary that (2.3) holds and there exists $a \in \mathrm{Jord}_\rho$ such that $l_1 < (a-1)/2 < l_2$.*

We now write a few consequences of the above discussion.

THEOREM 2.1: *Assume that (2.3) hold, $\mathrm{Jord}_\rho \cap [2l_1+1, 2l_2+1] = \emptyset$, and $l_1 \geq 0$. Then in the appropriate Grothendieck group it has an expansion of the form*

$$\delta \rtimes \sigma = \sigma_1 + \sigma_2 + Lang(\delta \rtimes \sigma),$$

where σ_1 and σ_2 are discrete series obtained from σ extending the triple of σ in the usual way as explained in [MT].

Proof: The discrete series σ_1 and σ_2 are constructed in the classification of discrete series [MT]. Next, Lemma 2.2 and Corollary 2.3 show that there is no non-tempered irreducible subquotient different from $Lang(\delta \rtimes \sigma)$. Note that $Lang(\delta \rtimes \sigma)$ comes with multiplicity one.

Lemma 2.1 shows that all other irreducible subquotients must be in discrete series. Therefore, the kernel of the equivariant morphism $\delta \rtimes \sigma \to \tilde{\delta} \rtimes \sigma$ must be a tempered representation of finite length having discrete series as irreducible subquotients. Since our groups have finite centers, they are projective objects in a category of all finite-length tempered representations. In particular, they appear as subrepresentations of $\delta \rtimes \sigma$. Hence all tempered irreducible subquotients of $\delta \rtimes \sigma$ are isomorphic to σ_1 or σ_2. Finally, ([M3], Theorem 2.3), for example, shows that they appear with multiplicity one in $\delta \rtimes \sigma$. This proves the theorem. ∎

The next result is well-known (see [T] and reference there). We include it here for the sake of completeness.

THEOREM 2.2: *Assume that $\rho \not\cong \tilde{\rho}$ or $2l_1 + 1 \notin \mathbb{Z}$; then $\delta \rtimes \sigma$ is irreducible.*

Proof: A well-known result of Tadić shows that $\delta \rtimes \sigma$ cannot have discrete series subquotients. (This also follows directly from the classification of discrete series.) Lemma 2.1 and Lemma 2.2 then show that the only possible irreducible subquotient of $\delta \rtimes \sigma$ is its Langlands quotient. ∎

Finally, we prove that in the remainder of the paper we may assume that (2.3) holds.

THEOREM 2.3: *Assume that $\rho \cong \widetilde{\rho}$ and $2l_1+1 \in \mathbb{Z}$ hold.*

(i) *If $\mathrm{Jord}_\rho \neq \emptyset$ but $2l_1+1-a \notin 2\mathbb{Z}$, $a \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible.*

(ii) *Assume $\mathrm{Jord}_\rho = \emptyset$. Then $\delta \rtimes \sigma$ is reducible if and only if $l_1 \geq -1/2$, and $2l_1+1$ is even if and only if $L(s,\rho,r)$ has a pole at $s=0$. If it is reducible, then in the appropriate Grothendieck group*

$$\delta \rtimes \sigma = \begin{cases} \sigma_1 + Lang(\delta \rtimes \sigma), & \text{if } l_1 = -1/2, \\ \sigma_1 + \sigma_2 + Lang(\delta \rtimes \sigma), & \text{if } l_1 \geq 0, \end{cases}$$

where σ_i, $i=1,2$, is a discrete series such that $\mathrm{Jord}(\sigma_i)$ is equal to

$$\begin{cases} \mathrm{Jord}(\sigma) \cup \{(2l_2+1,\rho)\}, & \text{if } l_1 = -1/2, \\ \mathrm{Jord}(\sigma) \cup \{(2l_1+1,\rho),(2l_2+1,\rho)\}, & \text{if } l_1 \geq 0, \end{cases}$$

and their ϵ_{σ_i}, $i=1,2$, are determined as follows. If $l_1 \geq 0$, they are two possible extensions of ϵ such that $\epsilon_{\sigma_i}(2l_1+1,\rho)\epsilon_{\sigma_i}(2l_2+1,\rho)^{-1}=1$, $i=1,2$. Moreover, σ_1 and σ_2 are non-isomorphic. If $l_1=-1/2$, then $\epsilon_{\sigma_i}(2l_2+1,\rho)^{-1}=1$.

Proof: This is a simple exercise combining Lemma 2.1, Lemma 2.2, Corollary 2.1 and the classification of discrete series recalled in the last section. In the case $l_1 \geq 0$, we also need to use the argument used in the proof of Theorem 2.1 to exibit all irreducible subquotients. ∎

3. Strongly positive discrete series I

Let $\delta \rtimes \sigma$ be the induced representation given by (2.1)–(2.3) and assuming that σ is a strongly positive discrete series. In this section we investigate the structure of $\delta \rtimes \sigma$ assuming

$$l_1 \leq -1.$$

It is more convenient to write $l_1 = -a-1$ and $l = l_2$. Thus, we write $\delta \rtimes \sigma$ as follows:

$$\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma,$$

and the assumptions (2.1)–(2.3) can be rephrased as follows: $a \in \mathbb{Z}_{\geq 0}$, $l-a \in \mathbb{Z}_{>0}$, and σ is a strongly positive discrete series, where $\mathrm{Jord}_\rho \neq \emptyset$ and $l-l' \in \mathbb{Z}$, for any $2l'+1 \in \mathrm{Jord}_\rho$.

Let us write $2a_0+1$ for the largest element of Jord_ρ such that $2a_0+1 \leq 2l+1$ if it exists. If $a_0 < l$, we define a strongly positive discrete series σ_0 such that the following hold (see Proposition 1.2):

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_0) = \mathrm{Jord}_{\rho'}, & \rho' \neq \rho \\ \mathrm{Jord}_\rho(\sigma_0) = \mathrm{Jord}_\rho \backslash \{2a_0+1\} \cup \{2l+1\}. \end{cases} \tag{3.1}$$

The main result of this section is the following proposition.

PROPOSITION 3.1:

(i) *Assume* $\mathrm{Jord}_\rho \cap [2a+1, 2l+1] \neq \emptyset$. *(Obviously,* $2a_0+1 \in \mathrm{Jord}_\rho \cap [2a+1, 2l+1]$.*) If* $a_0 = l$, *then the induced representation* $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ *is irreducible. Otherwise, in the appropriate Grothendieck group*

$$\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma = Lang(\delta([\nu^{a+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0) + Lang(\delta([\nu^{a+1}\rho, \nu^l\rho]) \rtimes \sigma).$$

In particular, $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ *has no tempered irreducible subquotients if* $a_0 > a$. *If* $a_0 = a$, *the above formula implies*

$$\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma = \sigma_0 + Lang(\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma).$$

(ii) *Assume* $\mathrm{Jord}_\rho \cap [2a+1, 2l+1] = \emptyset$. *Then* $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ *is irreducible.*

We prove the proposition in the more difficult case $2a+1 \in \mathrm{Jord}_\rho$. The case $2a+1 \notin \mathrm{Jord}_\rho$ goes the same way but it is easier.

The proposition is a consequence of the next three lemmas.

LEMMA 3.1: *Assume* $a_0 = l$. *Then the induced representation* $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ *is irreducible. In particular, Proposition 3.1 (ii) holds.*

Proof: First, it has no discrete series subquotient since it would have $(2l+1, \rho)$ twice in its set of Jordan blocks. Next, Lemma 2.1 shows that there is no tempered non-square integrable subquotient, or otherwise, according to (2.11), $2l_-+1 = (2l+1)_-$ would be defined and

$$\epsilon(2l+1,\rho) \cdot \epsilon(2l_-+1,\rho) = 1,$$

and this contradicts strong positivity of σ. Next, using the fact that σ is strongly positive, Lemma 2.2 shows that $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ has no other irreducible non-tempered subquotient different from its Langlands quotient or otherwise, using the notation of that lemma, $\beta_1 = l_1 = -a-1$, $\sigma_1 \cong \sigma$, and π' is tempered and an irreducible subquotient of $\delta([\nu^{a+1}\rho, \nu^l\rho]) \rtimes \sigma$, but as we just observed induced representation of that cannot have a tempered irreducible subquotient. ∎

LEMMA 3.2: *Assume $a_0 = a$. Then we have*

$$\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma = \sigma_0 + Lang(\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma).$$

In particular, the last assertion of Proposition 3.1 (i) holds.

Proof: We write $\mathrm{Jord}_\rho = \{2b_1+1, \ldots, 2b_k+1\}$. Assume $a = b_{i_0}$. Thus, our assumption $a_0 = a$ means $b_i < l$, for $i \leq i_0$, and $b_i > l$, for $i > i_0$. Let ϕ: $\mathrm{Jord}_\rho \to \mathrm{Jord}'(\sigma')_\rho$ be the increasing injection from the classification of strongly positive discrete series (see Section 2).

We put $2c_i + 1 = \phi(2b_i+1)$, $i = 1, \ldots, k$. We may also define strongly positive discrete series σ'' (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma'') = \mathrm{Jord}_{\rho'}, & \rho' \neq \rho, \\ \mathrm{Jord}_\rho(\sigma'') = \mathrm{Jord}_\rho(\sigma'). \end{cases}$$

We have

$$\sigma \hookrightarrow \delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{c_k+1}\rho, \nu^{b_k}\rho]) \rtimes \sigma''.$$

Now, using [Ze], we have the following chain of equivariant morphisms:

$$\begin{aligned}(3.2)\quad & \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \times \\ & \times \delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{c_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{c_k+1}\rho, \nu^{b_k}\rho]) \rtimes \sigma'' \cong \\ & \delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \times \delta([\nu^{c_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{c_k+1}\rho, \nu^{b_k}\rho]) \rtimes \sigma'' \cong \\ & \cdots \\ & \delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \times \delta([\nu^{c_{i_0}+1}\rho, \nu^{b_{i_0}}\rho]) \times \\ & \times \delta([\nu^{c_{i_0+1}+1}\rho, \nu^{b_{i_0+1}}\rho]) \times \times \cdots \times \delta([\nu^{c_k+1}\rho, \nu^{b_k}\rho]) \rtimes \sigma''.\end{aligned}$$

Since $a = b_{i_0}$, [Ze] implies

$$\delta([\nu^{c_{i_0}+1}\rho,\ \nu^l\rho]) \hookrightarrow \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \times \delta([\nu^{c_{i_0}+1}\rho, \nu^{b_{i_0}}\rho]).$$

Now, the last induced representation in (3.2) has also the following induced representation as a subrepresentation:

$$\begin{aligned}(3.3)\, & \delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{c_{i_0-1}+1}\rho, \nu^{b_{i_0-1}}\rho]) \times \delta([\nu^{c_{i_0}+1}\rho, \nu^l\rho]) \times \\ & \times \delta([\nu^{c_{i_0+1}+1}\rho, \nu^{b_{i_0+1}}\rho]) \times \cdots \times \delta([\nu^{c_k+1}\rho, \nu^{b_k}\rho]) \rtimes \sigma''.\end{aligned}$$

The classification of discrete series implies that the induced representation in (3.3) has the unique irreducible subrepresentation. This representation is σ_0.

We want to prove that this representation is the unique irreducible subrepresentation of the last induced representation in (3.2) and that it appears there as a subquotient with multiplicity one. To accomplish that it is enough to prove that

$$\delta([\nu^{c_1+1}\rho,\nu^{b_1}\rho])\otimes\cdots\otimes\delta([\nu^{a+1}\rho,\ \nu^{l}\rho])\otimes\delta([\nu^{c_{i_0}+1}\rho,\nu^{b_{i_0}}\rho])\otimes$$
$$\otimes\delta([\nu^{c_{i_0+1}+1}\rho,\nu^{b_{i_0+1}}\rho])\otimes\cdots\otimes\delta([\nu^{c_k+1}\rho,\nu^{b_k}\rho])\otimes\sigma''$$

appear in the appropriate Jacquet module of that induced representation with multiplicity one. Although this situation does not meet the assumption of [MT, Lemma 4.1] exactly, since we have

$$b_{i_0}<l<b_{i_0+1}$$

the proof of that lemma can be easily adapted. We leave the simple combinatorial verification to the reader.

Thus, we have proven that $\delta([\nu^{a+1}\rho,\ \nu^{l}\rho])\rtimes\sigma$ contains σ_0 as an irreducible subquotient with multiplicity one. Now, to complete the proof of the lemma, we observe that Lemma 2.1 implies that any tempered irreducible subquotient must be in a discrete series. Next, the definition of σ_0 given the above and Theorem 1.1 show that any possible discrete series subquotient of $\delta([\nu^{a+1}\rho,\ \nu^{l}\rho])\rtimes\sigma$ must have the same supercuspidal support as σ_0. Hence they must have the same set of Jordan blocks, as can be easily established from Theorem 1.1 (see [MT], Section 8 for the proof). If that discrete series subquotient would not be strongly positive, its triple would dominate an alternated triple having at most $\#\,\mathrm{Jord}(\sigma_0)_{\rho'}-2$ elements of the form (b,ρ'), for some ρ', as follows directly from the definition of an admissible triple given in Section 1. Now, counting the number of elements of the form (b,ρ') in that alternated triple, we arrive at a contradiction with the definition of an alternated triple since the triple of σ_0 is already alternated.

Finally, Lemma 2.2 shows that $\delta([\nu^{a+1}\rho,\ \nu^{l}\rho])\rtimes\sigma$ has no non-tempered irreducible subquotient other than its Langlands quotient. ∎

The next lemma completes the proof of Proposition 3.1 We remind the reader that we assume $2a+1\in\mathrm{Jord}_\rho$.

LEMMA 3.3: *Assume $a_0<l$. Then for any $2b+1\in\mathrm{Jord}_\rho\cap[2a+1,\ 2l+1[$, we have*

(3.4)
$$\delta([\nu^{b+1}\rho,\ \nu^{l}\rho])\rtimes\sigma=Lang(\delta([\nu^{b+1}\rho,\ \nu^{a_0}\rho])\rtimes\sigma_0)+Lang(\delta([\nu^{b+1}\rho,\ \nu^{l}\rho])\rtimes\sigma)$$

in the appropriate Grothendieck group.

Proof: This lemma will be proven by induction on a number of elements in $\mathrm{Jord}_\rho \cap\,]2b+1,\ 2l+1[$. First, if this intersection is empty, then $a_0 = b$ and the lemma follows from Lemma 3.2.

To prove the step of the induction, we slightly abuse notation assuming that the claim holds for all $2b+1 \in \mathrm{Jord}_\rho \cap\,]2a+1,\ 2l+1[$ and prove the claim holds for $2a+1$.

First, we define some auxiliary discrete series. Let $2c+1$ be in $\mathrm{Jord}_\rho \cap\, [2a+1,\ 2l+1[$. We use Proposition 1.2 to define a strongly positive discrete series σ_c

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_c) = \mathrm{Jord}_{\rho'}, & \rho' \not\simeq \rho, \\ \mathrm{Jord}_\rho(\sigma_c) = \mathrm{Jord}_\rho \backslash \{2c+1\} \cup \{2l+1\}. \end{cases}$$

We remind the reader that the assumption of Lemma 3.3, $a_0 < l$, implies that $2l+1 \notin \mathrm{Jord}_\rho$. Thus the discrete series σ_c is well-defined.

Now, since the triple of σ is alternated, Lemma 2.2 implies that any non-tempered irreducible subquotient of $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ different from its Langlands quotient has the following form:

$$Lang(\delta([\nu^{a+1}\rho,\ \nu^c\rho]) \rtimes \sigma_c),$$

for some $2c+1 \in \mathrm{Jord}_\rho \cap\,]2a+1,\ 2l+1[$. By the same lemma, we also must have that σ_c is an irreducible subquotient of $\delta([\nu^{c+1}\rho, \nu^l\rho]) \rtimes \sigma$. Applying now the inductive assumption this is possible only if $c = a_0$.

Thus, we have determined that $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ has

$$Lang(\delta([\nu^{a+1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)$$

as its only possible non-tempered irreducible subquotient other than its Langlands quotient $Lang(\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma)$.

Lemma 3.4: *$\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ contains $Lang(\delta([\nu^{a+1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)$ in its composition series with multiplicity one.*

Proof: To prove this we apply [Ze] to get

$$\begin{aligned} (3.5)\quad \delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma &\hookrightarrow \\ &\delta([\nu^{b+1}\rho,\ \nu^l\rho]) \times \delta([\nu^{a+1}\rho,\ \nu^b\rho]) \rtimes \sigma \hookrightarrow \\ &\delta([\nu^{a_0+1}\rho,\ \nu^l\rho]) \times \delta([\nu^{b+1}\rho,\ \nu^{a_0}\rho]) \times \delta([\nu^{a+1}\rho,\ \nu^b\rho]) \rtimes \sigma. \end{aligned}$$

Next, Lemma 3.2 implies that the last induced representation in (3.5) contains

$$Lang(\delta([\nu^{a+1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)$$

as an irreducible subquotient. Further, by definition, we have

$$Lang(\delta([\nu^{a+1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0) \hookrightarrow \delta([\nu^{-a_0}\rho,\ \nu^{-a-1}\rho]) \rtimes \sigma_0.$$

Thus, by Frobenius reciprocity,

$$\mu^*(Lang(\delta([\nu^{a+1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)) \geq \delta([\nu^{-a_0}\rho,\ \nu^{-a-1}\rho]) \otimes \sigma_0.$$

Hence, to complete the proof of Lemma 3.4, we show that the multiplicity of

$$\delta([\nu^{-a_0}\rho,\ \nu^{-a-1}\rho]) \otimes \sigma_0$$

in the appropriate Jacquet modules of

$$\delta([\nu^{a+1}\rho,\ \nu^{l}\rho]) \rtimes \sigma$$

and

$$\delta([\nu^{a_0+1}\rho,\ \nu^{l}\rho]) \times \delta([\nu^{b+1}\rho,\ \nu^{a_0}\rho]) \times \delta([\nu^{a+1}\rho,\ \nu^{b}\rho]) \rtimes \sigma$$

is one.

We prove multiplicity one only in

$$\mu^*(\delta([\nu^{a_0+1}\rho,\ \nu^{l}\rho]) \times \delta([\nu^{b+1}\rho,\ \nu^{a_0}\rho]) \times \delta([\nu^{a+1}\rho,\ \nu^{b}\rho]) \rtimes \sigma).$$

The other one is analogous.

So, let $\mu^*(\sigma) \geq \delta \otimes \sigma_1$ be an irreducible constituent, and indices $0 \leq j \leq i \leq l - a_0$, $0 \leq j' \leq i' \leq a_0 - b$, $0 \leq j'' \leq i'' \leq b - a$. We have the following estimates:

$$\begin{aligned}(3.6)\quad \delta([\nu^{-a_0}\rho,\nu^{-a-1}\rho]) \leq\ & \delta([\nu^{i-l}\rho,\nu^{-a_0-1}\rho]) \times \delta([\nu^{l+1-j}\rho,\nu^{l}\rho])\times \\ & \delta([\nu^{i'-a_0}\rho,\nu^{-b-1}\rho]) \times \delta([\nu^{a_0+1-j'}\rho,\nu^{a_0}\rho])\times \\ & \delta([\nu^{i''-b}\rho,\nu^{-a-1}\rho]) \times \delta([\nu^{b+1-j''}\rho,\nu^{b}\rho]) \times \delta\end{aligned}$$

and

(3.7)

$$\sigma_0 \leq \delta([\nu^{l+1-i}\rho,\nu^{l-j}\rho]) \times \delta([\nu^{a_0+1-i'}\rho,\nu^{a_0-j'}\rho]) \times \delta([\nu^{b+1-i''}\rho,\nu^{b-j''}\rho]) \rtimes \sigma_1.$$

We see from (3.6) that $j = j' = j'' = 0$, δ is trivial and $\sigma_1 = \sigma$. Next, since $i'' - b \geq -b$, $i' - a_0 \geq -a_0$, we see that $i' = i'' = 0$ and $i = l - a_0$. Now, we see that the left-hand side of (3.6) is contained in the right-hand side with multiplicity one. Also, (3.7) reduces to

$$\sigma_0 \leq \delta([\nu^{a_0+1}\rho,\nu^{l}\rho]) \rtimes \sigma,$$

and Lemma 3.2 shows that the left-hand side is contained in the right-hand side with multiplicity one. ∎

The next lemma completes the proof of Lemma 3.3.

LEMMA 3.5: *The induced representation $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ does not contain tempered irreducible subquotients.*

Proof: First, we can apply Lemma 2.1 directly to see that $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ has no tempered irreducible subquotients that are not in discrete series. Next, the definition of σ_a given above and Theorem 1.1 show that any possible discrete series subquotient of $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$ must have the same supercuspidal support as σ_a. Hence they must have the same set of Jordan blocks, as can be easily established from Theorem 1.1 (see [MT], Section 8 for the proof). If that discrete series subquotient would not be strongly positive, its triple would dominate an alternated triple having at most $\#\,\mathrm{Jord}(\sigma_a)_{\rho'} - 2$ elements of the form (b, ρ'), for some ρ', as follows directly from the definition of an admissible triple given in Section 1. Now, counting the number of elements of the form (b, ρ') in that alternated triple, we arrive at a contradiction with the definition of an alternated triple since the triple of σ_a is already alternated.

Now, Proposition 1.2 shows that σ_a is the only possible discrete series subquotient of $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$. We use Theorem 1.2 to show that σ_a is not a subquotient of $\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma$.

So, let $\mu^*(\sigma) \geq \delta \otimes \sigma_1$ be an irreducible constituent, and indices $0 \leq j \leq i \leq l-a$. We have the following terms in $\mu^*(\delta([\nu^{a+1}\rho,\ \nu^l\rho]) \rtimes \sigma)$:

$$\delta([\nu^{i-l}\rho, \nu^{-a-1}\rho]) \times \delta([\nu^{l+1-j}\rho, \nu^l\rho]) \times \delta \otimes \delta([\nu^{l+1-i}\rho, \nu^{l-j}\rho]) \rtimes \sigma_1. \tag{3.8}$$

The goal is to find all terms that might produce $\delta([\nu^{a+1}\rho, \nu^b\rho]) \otimes \sigma_b$, since we have an obvious inclusion

$$\sigma_a \hookrightarrow \delta([\nu^{a+1}\rho,\ \nu^b\rho]) \rtimes \sigma_b$$

that follows from the definition of those discrete series (see Theorem 1.1 (ii)). The first part of the above tensor product in (3.8) shows that $i = l - a$. We also have $j = 0$, or otherwise $\nu^l\rho \in [\nu^{a+1}\rho,\ \nu^b\rho]$, which is a contradiction, since by the assumption of Lemma 3.3

$$2b + 1 \leq 2a_0 + 1 < 2l + 1.$$

Now, we see that $\delta \cong \delta([\nu^{a+1}\rho,\ \nu^b\rho])$. This means that there exists an irreducible representation σ'' such that

$$\sigma \hookrightarrow \nu^b\rho \times \cdots \times \nu^{a+1}\rho \rtimes \sigma''.$$

Next, since σ is strongly positive, σ'' is also strongly positive (see Proposition 1.1). Now, comparing Jordan blocks as in ([MT], Section 8) we see that

$$\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma'') \setminus \{(2a+1,\rho)\} \cup \{(2b+1,\rho)\}.$$

In particular, $(2a+1,\rho) \notin \mathrm{Jord}(\sigma)$. This contradicts our assumption. ∎

4. Strongly positive discrete series II

Let $\delta \rtimes \sigma$ be the induced representation given by (2.1)–(2.3) and assume that σ is a strongly positive discrete series. In this section we investigate the structure of $\delta \rtimes \sigma$ assuming

$$l_1 \geq 0.$$

Theorem 2.1 enables us to make the following assumption:

$$\mathrm{Jord}_\rho \cap [2l_1+1,\ 2l_2+1] \neq \emptyset. \tag{4.1}$$

Before we state the main result of this section, let us define some strongly positive discrete series. Let

$$2a_0+1 = \max \mathrm{Jord}_\rho \cap [2l_1+1,\ 2l_2+1]$$

and

$$2b_0+1 = \min \mathrm{Jord}_\rho \cap [2l_1+1,\ 2l_2+1].$$

First, σ_0 is a strongly positive discrete series obtained from σ using (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_0) = \mathrm{Jord}_{\rho'}, & \rho' \not\cong \rho, \\ \mathrm{Jord}_\rho(\sigma_0) = \mathrm{Jord}_\rho \setminus \{(2a_0+1,\rho)\} \cup \{(2l_2+1,\rho)\}. \end{cases}$$

Proposition 3.1 (i) implies

$$\sigma_0 \hookrightarrow \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

Next, σ_1 is a strongly positive discrete series obtained from σ using (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_1) = \mathrm{Jord}_{\rho'}, & \rho' \not\cong \rho, \\ \mathrm{Jord}_\rho(\sigma_1) = \mathrm{Jord}_\rho \setminus \{(2b_0+1,\rho)\} \cup \{(2l_1+1,\rho)\}. \end{cases}$$

Again Proposition 3.1 (i) implies

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1.$$

Finally, σ_{10} is a strongly positive discrete series obtained from σ_1 using (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_{10}) = \mathrm{Jord}_{\rho'}(\sigma_1), \quad \rho' \not\cong \rho, \\ \mathrm{Jord}_{\rho}(\sigma_{10}) = \mathrm{Jord}_{\rho}(\sigma_1) \setminus \{(2a_0+1, \rho)\} \cup \{(2l_2+1, \rho)\}. \end{cases}$$

Moreover, Proposition 3.1 (i) implies

$$\sigma_{10} \hookrightarrow \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1.$$

The main result of this section is the following theorem.

THEOREM 4.1: *Under the above assumptions, we have the following.*

(i) *If $2l_1+1, 2l_2+1 \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible.*

(ii) *Assume $2l_1+1 \in \mathrm{Jord}_\rho$ and $2l_2+1 \notin \mathrm{Jord}_\rho$. Then, if $a_0 = l_1$ (that is, $\mathrm{Jord}_\rho \cap \,]2l_1+1,\ 2l_2+1[\, = \emptyset$), then in an appropriate Grothendieck group*

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + \sigma_{temp},$$

where σ_{temp} is a unique common irreducible subquotient of

$$\delta \rtimes \sigma$$

and

$$\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_0.$$

If $a_0 > l_1$, then in an appropriate Grothendieck group

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0).$$

(iii) *Assume $2l_1+1 \notin \mathrm{Jord}_\rho$ and $2l_2+1 \in \mathrm{Jord}_\rho$. Then, if $b_0 = l_2$ (that is, $\mathrm{Jord}_\rho \cap \,]2l_1+1,\ 2l_2+1[\, = \emptyset$), then in an appropriate Grothendieck group*

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + \sigma_{temp},$$

where σ_{temp} is a unique common irreducible subquotient of

$$\delta \rtimes \sigma$$

and

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1.$$

If $b_0 < l_2$, then in an appropriate Grothendieck group

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1).$$

(iv) *Assume* $2l_1+1 \notin \mathrm{Jord}_\rho$ *and* $2l_2+1 \notin \mathrm{Jord}_\rho$. *We have two cases.*
First, if $b_0 < a_0$, *then in an appropriate Grothendieck group*

$$\begin{aligned}\delta \rtimes \sigma =& Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0) \\ &+ Lang(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1) + Lang(\delta([\nu^{-b_0}\rho, \nu^{a_0}\rho]) \rtimes \sigma_{10}).\end{aligned}$$

Next, if $b_0 = a_0$, *then we define two discrete series and* σ^i, $i = 0, 1$ *as follows. The triple* $(\mathrm{Jord}^i, \sigma', \epsilon^i)$ *of* σ^i *triple is defined as follows.*

- $\mathrm{Jord}^0 = \mathrm{Jord}(\sigma_0) \cup \{(2l_1+1, \rho), (2a_0+1, \rho)\}$ *and*

$$\mathrm{Jord}^1 = \mathrm{Jord}(\sigma_1) \cup \{(2a_0+1, \rho), (2l_2+1, \rho)\}.$$

- ϵ^i *is the unique extension of* ϵ_{σ_i} *to* Jord^i *such that*

$$\begin{cases} \epsilon^0(2a_0+1, \rho) \cdot \epsilon^0(2l_1+1, \rho)^{-1} = 1 \\ \epsilon^1(2l_2+1, \rho) \cdot \epsilon^1(2a_0+1, \rho)^{-1} = 1 \end{cases}$$

and all other products $\epsilon^i(2a+1, \rho) \cdot \epsilon^i((2a+1)_-, \rho)^{-1} = -1$, $i = 0, 1$.
Now, in an appropriate Grothendieck group

$$\begin{aligned}\delta \rtimes \sigma =& Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0) \\ &+ Lang(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1) + \sigma^0 + \sigma^1.\end{aligned}$$

The next couple of lemmas will prove the theorem. We start this section with a technical result.

Lemma 4.1: *If* $\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^b\rho]) \otimes \sigma''$, *where* $b \in \mathbb{R}$, $b - l_1 \in \mathbb{Z}_{>0}$, *and* σ'' *is irreducible, then* $b = b_0$ *and* $\sigma'' \cong \sigma_1$ *(defined in (ii) above).*

Proof: First, $\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^b\rho]) \otimes \sigma''$ implies that there exists an irreducible representation ${\sigma''}_1$ such that

$$\sigma \hookrightarrow \nu^b\rho \times \cdots \times \nu^{l_1+1} \rtimes {\sigma''}_1. \tag{4.2}$$

Since σ is strongly positive, ${\sigma''}_1$ is also strongly positive (see Proposition 1.1). We also see that (using methods of [MT], Section 8)

$$\begin{cases} \mathrm{Jord}_{\rho'}({\sigma''}_1) = \mathrm{Jord}_{\rho'}, & \rho' \not\cong \rho, \\ \mathrm{Jord}_\rho({\sigma''}_1) = \mathrm{Jord}_\rho \setminus \{2b+1\} \cup \{2l_1+1\}. \end{cases} \tag{4.3}$$

This determines ${\sigma''}_1$ completely (see Proposition 1.2). We now show that in fact $\sigma \hookrightarrow \delta([\nu^{l_1+1}, \nu^b\rho]) \rtimes {\sigma''}_1$.

If not, we take the smallest $k \geq 1$ such that there exists a sequence $b > a_1 > \cdots > a_k > l_1 + 1$ such that

$$\sigma \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{b}]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \times \cdots \times \delta([\nu^{l_1+1}\rho, \nu^{a_k}\rho]) \rtimes \sigma''_1. \tag{4.4}$$

Now, the assumption on minimality of k implies that we can permute essentially square integrable representations in (4.4) as we want, and the inclusion is still preserved. In particular, we have

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{a_k}\rho]) \\ \times \delta([\nu^{a_1+1}\rho, \nu^{b}]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \times \cdots \times \delta([\nu^{a_k+1}\rho, \nu^{a_{k-1}}\rho]) \rtimes \sigma''_1.$$

This implies $\sigma \hookrightarrow \nu^{a_k} \rtimes \sigma'''$, for some irreducible representation σ'''. Hence ([Mœ] Remark 5.1.2) implies $2a_k + 1 \in \mathrm{Jord}_\rho$. Since we have $b > a_k > l_1$, (4.3) implies that $2a_k + 1 \in \mathrm{Jord}_\rho(\sigma''_1)$. Now, Proposition 3.1 (ii) shows that $\delta([\nu^{l_1+1}\rho, \nu^{a_k}\rho]) \rtimes \sigma''_1$ is irreducible. In particular,

$$\delta([\nu^{-a_k}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma''_1 \cong \delta([\nu^{l_1+1}\rho, \nu^{a_k}\rho]) \rtimes \sigma''_1. \tag{4.5}$$

Combinig now (4.4) and (4.5) we obtain

$$\sigma \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{b}]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \times \cdots \times \delta([\nu^{-a_k}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma''_1.$$

Since [Ze] implies

$$\delta([\nu^{a_i+1}\rho, \nu^{a_{i-1}}]) \times \delta([\nu^{-a_k}\rho, \nu^{-l_1-1}\rho]) \cong \delta([\nu^{-a_k}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{a_i+1}\rho, \nu^{a_{i-1}}])$$

(here $a_0 = b$), we obtain

$$\sigma \hookrightarrow \delta([\nu^{-a_k}\rho, \nu^{-l_1-1}\rho]) \\ \times \delta([\nu^{a_1+1}\rho, \nu^{b}]) \times \delta([\nu^{a_2+1}\rho, \nu^{a_1}]) \times \cdots \times \delta([\nu^{a_k+1}\rho, \nu^{a_{k-1}}\rho]) \rtimes \sigma''_1.$$

Thus, for some irreducible representation σ''' we have

$$\sigma \hookrightarrow \nu^{-l_1-1}\rho \rtimes \sigma'''.$$

This contradicts a square-integrable criterion. Thus, we have shown that

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}, \nu^{b}\rho]) \rtimes \sigma''_1.$$

Applying Proposition 3.1 (i) we see that $b = b_0$. Now, $\sigma''_1 \cong \sigma_1$. Finally, a computation of Jacquet modules given in ([M3], Theorem 2.3) shows that $\sigma'' \cong \sigma_1$. The lemma is proven. ∎

Now, we are ready to prove Theorem 4.1 (i).

LEMMA 4.2: *If $2l_1+1, 2l_2+1 \in \mathrm{Jord}_\rho$, then $\delta \rtimes \sigma$ is irreducible.*

Proof: First, Lemma 2.1 (using notation there) implies that tempered subquotients would have a discrete series part σ_2 such that its set of Jordan blocks is obtained from σ removing $(2l_1+1,\rho)$ and $(2l_2+1,\rho)$. Since σ is strongly positive, a simple counting argument (using results recalled in Section 1) shows that is not possible.

This induced representation has no discrete series subquotient since such a subquotient would have, for example, $(2l_1+1,\rho)$ in its set of Jordan blocks.

We finally show that $\delta \rtimes \sigma$ does not have non-tempered irreducible subquotients other than its Langlands quotient. This completes the proof of irreducibility. So, let π be a non-tempered irreducible subquotient of $\delta \rtimes \sigma$. Then, by Lemma 2.2, we can find an irreducible representation π', and $\alpha, \beta \in \mathbb{R}$, $\alpha+\beta \in \mathbb{Z}_{\geq 0}$, $\beta-\alpha < 0$, such that

$$\pi \hookrightarrow \delta([\nu^{-\alpha}\rho,\ \nu^{\beta}\rho]) \rtimes \pi'.$$

According to Lemma 2.2, we have two cases.

If $\beta > l_1$, then σ_1 from that Lemma 2.2 is in fact the one defined in Theorem 3.1 (iii). Hence $2l_1+1$ appears twice in $\mathrm{Jord}_\rho(\sigma_1)$, and this is a contradiction.

If $\beta = l_1$, then $2\alpha+1 \in \mathrm{Jord}_\rho$, and if $\alpha < l_2$, π' is a tempered subquotient of $\delta([\nu^{\alpha_1+1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma$. This contradicts Proposition 3.1 (ii). ∎

The next lemma will help us determine possible irreducible subquotients in the proof of Theorem 4.1 (ii) and (iii).

LEMMA 4.3:

(i) *Assume $2l_1+1 \in \mathrm{Jord}_\rho$ and $2l_2+1 \notin \mathrm{Jord}_\rho$. Then $\delta \rtimes \sigma$ contains a tempered irreducible subquotient if and only if $\mathrm{Jord}_\rho \cap\,]2l_1+1,\ 2l_2+1[\, = \emptyset$. It is a unique common irreducible subquotient of $\delta \rtimes \sigma$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_0$. (Note that $a_0 = b_0 = 2l_1+1$.) If $\mathrm{Jord}_\rho \cap\,]2l_1+1,\ 2l_2+1[\, = \emptyset$, there is no non-tempered irreducible subquotient other than $Lang(\delta \rtimes \sigma)$.*

(ii) *Assume $2l_1+1 \notin \mathrm{Jord}_\rho$ and $2l_2+1 \in \mathrm{Jord}_\rho$. Then $\delta \rtimes \sigma$ contains a tempered irreducible subquotient if and only if $\mathrm{Jord}_\rho \cap\,]2l_1+1,\ 2l_2+1[\, = \emptyset$. It is a unique common irreducible subquotient of $\delta \rtimes \sigma$ and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$. (Note that $a_0 = b_0 = 2l_2+1$.) If $\mathrm{Jord}_\rho \cap\,]2l_1+1,\ 2l_2+1[\, = \emptyset$, there is no non-tempered irreducible subquotient other than $Lang(\delta \rtimes \sigma)$.*

Proof: We prove (ii). The proof of (i) is analogous but simpler. Thus, $2l_1+1 \notin \mathrm{Jord}_\rho$ and $2l_2+1 \in \mathrm{Jord}_\rho$. Our assumption and Lemma 2.1 (using notation of that lemma) show that all tempered irreducible subquotients are tempered of the form

$$\pi \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1,$$

where σ_1 is defined in Lemma 2.1. It seems that we slightly abuse notation since σ_1 is also defined differently at the beginning of this section, but we show that they are equal. In fact, (2.11) and Lemma 4.1 show that $\mathrm{Jord}_\rho \cap]2l_1+1,\ 2l_2+1[= \emptyset$. This proves the necessary condition for the existence of a tempered irreducible subquotient as well as that those two definitions of σ_1 are the same.

Obviously, as before, $\delta \rtimes \sigma$ cannot contain a discrete series subquotient. Also, Lemma 2.2 shows that there is no non-tempered irreducible subquotient other than $Lang(\delta \rtimes \sigma)$.

Finally, the existence of the tempered irreducible subquotient follows from the next lemma. ∎

LEMMA 4.4: *Assume $2l_1 + 1 \notin \mathrm{Jord}_\rho$ or $2l_2 + 1 \notin \mathrm{Jord}_\rho$ but not both, and $\mathrm{Jord}_\rho \cap]2l_1 + 1,\ 2l_2 + 1[= \emptyset$. Then $\delta \rtimes \sigma$ is reducible.*

Proof: Assume that $\delta \rtimes \sigma$ is irreducible. Hence

$$\delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

and we have the following computation:

$$\begin{aligned}(4.6)\quad \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma &\hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \\ &\quad \times\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma.\end{aligned}$$

Now, we show that for any irreducible subrepresentation $\pi \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$, the multiplicity of the irreducible representation

$$(4.7)\qquad \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \pi$$

in

$$\mu^*(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma)$$

is one. We apply Theorem 1.2 three times. Thus, for some irreducible constituent $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$, and indices $0 \leq j \leq i \leq l_2 - l_1$, $0 \leq j_1 \leq i_1 \leq l_2 - l_1$, and $0 \leq j_2 \leq i_2 \leq 2l_1 + 1$, we must have

$$\begin{cases} \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \\ \leq \delta([\nu^{i-l_2}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{i_1-l_2}\rho, \nu^{-l_1-1}\rho]) \\ \times\delta([\nu^{l_2+1-j_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{i_2-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1-j_2}\rho, \nu^{l_1}\rho]) \times \delta'', \\ \pi \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_1+1-i}\rho, \nu^{l_1-j}\rho]) \times \sigma''. \end{cases}$$

We see that $i = i_1 = l_2 - l_1$, $i_2 = 2l_1 + 1$, and $j_2 = 0$. Note that δ'' must be nondegenerate, so if $l_2 + 1 - j > l_1 + 1$ or $l_2 + 1 - j_1 > l_1 + 1$, then [Ze] implies

$$\begin{aligned}\delta &\cong \delta([\nu^{l_1+1}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2-j_1}\rho]) \\ &\cong \delta([\nu^{l_1+1}\rho, \nu^{l_2-j}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2-j_1}\rho]).\end{aligned}$$

This would imply $2(l_2 - j) + 1 \in \mathrm{Jord}_\rho$ or $2(l_2 - j_1) + 1 \in \mathrm{Jord}_\rho$ and this would contradict $\mathrm{Jord}_\rho \cap \,]2l_1 + 1,\ 2l_2 + 1[\, = \emptyset$.

Thus, $j = j_1 = l_2 - l_1$. Now, δ'' is trivial and hence $\sigma'' \cong \sigma$. Moreover, the second displayed formula reads

$$\pi \leq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma.$$

It is a well-known result of Harish-Chandra that π is contained there with multiplicity one. This proves the assertion.

Now, we analyze (4.6) in the next two cases.

• $2l_1 + 1 \in \mathrm{Jord}_\rho$ and $2l_2 + 1 \notin \mathrm{Jord}_\rho$. By the definition (see Section 1) this is equivalent to $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ is irreducible, and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is reducible. Now, $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is a direct sum of two non-equivalent tempered representations both of them being a subrepresentation of the right-hand side of (4.6). Hence Frobenius reciprocity implies that (4.7) with $\pi = \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ appears more than once in the appropriate Jacquet module. This is a contradiction.

• $2l_1 + 1 \notin \mathrm{Jord}_\rho$ and $2l_2 + 1 \in \mathrm{Jord}_\rho$. By the definition (see Section 1) this is equivalent to $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ is reducible, and $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is irreducible. Now, $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ is a direct sum of two non-equivalent tempered representations, say π_1 and π_2. Using the same methods as above we can show

$$\mu^*(\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \otimes \pi_i, \quad i = 1, 2.$$

Now, this combined with the fact that the right-hand side of (4.6) is a direct sum of two representations

$$\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes (\pi_1 \oplus \pi_2)$$

suffices to see that $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ must intersect them both non-trivialy. Hence $\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is reducible. This is a contradiction. ∎

Now, we prove the rest of Theorem 4.1 (ii).

LEMMA 4.5: *Assume* $2l_1+1 \in \mathrm{Jord}_\rho$ *and* $2l_2+1 \notin \mathrm{Jord}_\rho$, *and*

$$\mathrm{Jord}_\rho \cap \,]2l_1+1,\ 2l_2+1[\, \neq \emptyset.$$

Then we have

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0),$$

where σ_0 *is a strongly positive discrete series defined in Theorem 4.1 (ii).*

Proof: We first list all possible irreducible subquotients. Lemma 4.3 shows that there is no tempered irreducible subquotient. We investigate all possible non-tempered irreducible subquotients different from $Lang(\delta \rtimes \sigma)$. So, let π be such an irreducible subquotient of $\delta \rtimes \sigma$. Then, by Lemma 2.2, we can find an irreducible representation π', and $\alpha, \beta \in \mathbb{R}$, $\alpha+\beta \in \mathbb{Z}_{\geq 0}$, $\beta - \alpha < 0$, such that

$$\pi \hookrightarrow \delta([\nu^{-\alpha}\rho,\ \nu^{\beta}\rho]) \rtimes \pi'.$$

Also, according to Lemma 2.2, we have two cases.

If $\beta > l_1$, then we arrive at a contradiction as we did in the proof of Lemma 4.2. Thus, $\beta = l_1$. Then $2\alpha+1 \in \mathrm{Jord}_\rho$, $\alpha < l_2$, π' is a tempered subquotient of $\delta([\nu^{\alpha+1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma$ (see Lemma 2.2). According to Proposition 3.1 this is possible if and only if $\alpha = a_0$ and $\pi' = \sigma_0$. Thus, we have

$$\pi \hookrightarrow \delta([\nu^{-a_0}\rho,\ \nu^{l_1}\rho]) \rtimes \sigma_0.$$

Thus, $\pi \cong Lang(\delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)$. To complete the proof of the lemma we need to show that such π appears in $\delta \rtimes \sigma$ with multiplicity one.

To finish this, we observe that

$$\begin{cases} \delta([\nu^{-l_1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{a_0+1}\rho,\ \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma, \\ Lang(\delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0) \leq \delta([\nu^{a_0+1}\rho,\ \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma. \end{cases}$$

To complete the proof, we need to show that the multiplicity of

$$\delta([\nu^{-a_0}\rho,\ \nu^{l_1}\rho]) \otimes \sigma_0$$

in

$$\mu^*(\delta([\nu^{-l_1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma)$$

and

$$\mu^*(\delta([\nu^{a_0+1}\rho,\ \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma)$$

is one since, as before,

$$\mu^*(Lang(\delta([\nu^{-l_1}\rho,\ \nu^{a_0}\rho]) \rtimes \sigma_0)) \geq \delta([\nu^{-a_0}\rho,\ \nu^{l_1}\rho]) \otimes \sigma_0,$$

by Frobenius reciprocity.

The computation of such multiplicities is completely analogous to that given in Lemma 3.4, and consequently is omitted. ∎

Now, we prove the rest of Theorem 4.1 (iii).

LEMMA 4.6: *Assume* $2l_1 + 1 \notin \mathrm{Jord}_\rho$, $2l_2 + 1 \in \mathrm{Jord}_\rho$, *and*

$$\mathrm{Jord}_\rho \cap \,]2l_1 + 1,\ 2l_2 + 1[\, \neq \emptyset.$$

Then in an appropriate Grothendieck group

$$\delta \rtimes \sigma = Lang(\delta \rtimes \sigma) + Lang(\delta([\nu^{l_2}\rho, \nu^{-b_0}\rho]) \rtimes \sigma_1),$$

where σ_1 *is a strongly positive discrete series defined in Theorem 4.1 (iii).*

Proof: First, there is no tempered irreducible subquotient by Lemma 4.3. Now, we investigate all possible non-tempered irreducible subquotients different from $Lang(\delta \rtimes \sigma)$. So, let π be such an irreducible subquotient of $\delta \rtimes \sigma$. Then, by Lemma 2.2, we can find an irreducible representation π', and $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta \in \mathbb{Z}_{\geq 0}$, $\beta - \alpha < 0$, such that

$$\pi \hookrightarrow \delta([\nu^{-\alpha}\rho,\ \nu^{\beta}\rho]) \rtimes \pi'. \tag{4.8}$$

Also, according Lemma 2.2, we have two cases.

If $\beta = l_1$, then $2\alpha + 1 \in \mathrm{Jord}_\rho$, $\alpha < l_2$, π' is a tempered subquotient of $\delta([\nu^{\alpha+1}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma$ (see Lemma 2.2). This contradicts Proposition 3.1 (ii).

If $\beta > l_1$, then Lemma 4.1 implies that σ_1 is just the one defined in Theorem 4.1 (iii) and $\beta = b_0$. Now, Lemma 2.2 and Proposition 3.1 (ii) imply that

$$\pi' \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{-\alpha-1}\rho]) \rtimes \sigma.$$

Combining this with (4.8) we arrive at

$$\pi \hookrightarrow \delta([\nu^{-\alpha}\rho,\ \nu^{b_0}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-\alpha-1}\rho]) \rtimes \sigma.$$

Using this and [Ze] we obtain the following equivariant maps:

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-\alpha}\rho,\ \nu^{b_0}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{-\alpha-1}\rho]) \\ &\to \delta([\nu^{-l_2}\rho, \nu^{-\alpha-1}\rho]) \times \delta([\nu^{-\alpha}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma.\end{aligned}$$

The kernel of the second one is isomorphic to $\delta([\nu^{-l_2}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma$. We claim

$$\pi \hookrightarrow \delta([\nu^{-l_2}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma,$$

and this implies

$$\pi \cong Lang(\delta([\nu^{-b_0}\rho,\ \nu^{l_2}\rho]) \rtimes \sigma). \tag{4.9}$$

Otherwise, we would have $\pi \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{-\alpha-1}\rho]) \rtimes \pi''$. We may analyze this using Lemma 2.2. In particular, we have $-\alpha - 1 \geq l_1$. This is a contradiction. Thus, (4.9) holds. To complete the proof of the lemma we need to show that such π (defined by (4.9)) appears in $\delta \rtimes \sigma$ with multiplicity one.

To finish this, we observe that

$$\begin{cases} \delta([\nu^{-l_2}\rho,\ \nu^{l_1}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{-l_2}\rho,\ \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma_1. \\ \pi \leq \delta([\nu^{-l_2}\rho,\ \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma_1. \end{cases} \tag{4.10}$$

To complete the proof, we show that the multiplicity of $\delta([\nu^{-l_2}\rho,\ \nu^{b_0}\rho]) \otimes \sigma_1$ in appropriate Jacquet modules of the left-hand side and right-hand side of the first formula in (4.10) is equal to one using methods described in the proof of Lemma 3.4. The details are left to the reader. ∎

Finally, we come to the proof of Theorem 4.1 (iv). We split the proof into several lemmas. In the first we determine all non-tempered irreducible subquotients in $\delta \rtimes \sigma$.

Lemma 4.7: *All non-tempered irreducible subquotients of $\delta \rtimes$ are given by*

$$Lang(\delta \rtimes \sigma), \quad Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0), \quad Lang(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1),$$

and if $a_0 > b_0$ we have also $Lang(\delta([\nu^{-b_0}\rho, \nu^{a_0}\rho) \rtimes \sigma_{10})$. They all appear with multiplicity one.

Proof: It is well-known that $Lang(\delta \rtimes \sigma)$ appears with multiplicity one. For the other two

$$Lang(\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0)$$

and

$$Lang(\delta([\nu^{-b_0}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1)$$

we may proceed exactly as in the proofs of Lemma 4.5 and Lemma 4.6, respectively.

It remains to analyze $a_0 > b_0$ and $Lang(\delta([\nu^{-b_0}\rho, \nu^{a_0}\rho]) \rtimes \sigma_{10})$. This follows from the fact that (as we remarked in the previous two lemmas; see the text below (4.10))

$$\delta([\nu^{-l_2}\rho,\ \nu^{-a_0-1}\rho]) \times \delta([\nu^{-a_0}\rho,\ \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho,\ \nu^{b_0}\rho]) \rtimes \sigma_1 \geq \delta([\nu^{-l_2}\rho,\ \nu^{l_1}\rho]) \rtimes \sigma,\ Lang(\delta([\nu^{-b_0}\rho, \nu^{a_0}\rho]) \rtimes \sigma_{10}).$$

We leave the simple verification to the reader. ∎

To complete the proof of Theorem 4.1 (iv) we prove

LEMMA 4.8: *Assume that $2l_1 + 1, 2l_2 + 1 \notin \mathrm{Jord}_\rho$. If $b_0 < a_0$, then there is no tempered irreducible subquotient in $\delta \rtimes \sigma$. Otherwise, $\delta \rtimes \sigma$ contains σ^0 and σ^1 as the unique irreducible tempered irreducible subquotients each with multiplicity one.*

Proof: We first analyze possible tempered irreducible subquotients. Lemma 2.1 shows that we cannot have tempered irreducible subquotients that are not square integrable. A discrete series subquotient must have its Jordan blocks of the form

$$\mathrm{Jord}(\sigma) \cup \{(2l_1 + 1, \rho),\ (2l_2 + 1, \rho)\}.$$

So, let π be a discrete series which is a subquotient of $\delta \rtimes \sigma$. We need to compute its partial ϵ_π function. First, since σ is a strongly positive discrete series, there must be $(2a + 1, \rho) \in \mathrm{Jord}(\pi)$ such that $(2a + 1)_- := 2a_- + 1$ is defined, and

$$\pi \hookrightarrow \delta([\nu^{-a_-}\rho, \nu^{a}\rho]) \rtimes \pi',$$

where π' is a discrete series such that

$$\mathrm{Jord}(\pi') = \mathrm{Jord}(\pi) \setminus \{(2a + 1, \rho), (2a_- + 1, \rho)\},$$

and its $\epsilon_{\pi'}$ is the restriction of ϵ_π of π to $\mathrm{Jord}(\pi')$. Now, Frobenius reciprocity implies that

$$\delta([\nu^{-a_-}\rho, \nu^{a}\rho]) \otimes \pi' \leq \mu^*(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma).$$

We analyze this using Theorem 1.2. So, let $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$ be an irreducible constituent, and indices $0 \leq j \leq i \leq l_1 + l_2 + 1$. We have the following formulae:

$$(4.11)\qquad \begin{cases} \delta([\nu^{-a_-}\rho, \nu^{a}\rho]) \leq \delta([\nu^{i-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_2+1-j}\rho, \nu^{l_2}\rho]) \times \delta'', \\ \pi' \leq \delta([\nu^{l_2+1-i}\rho, \nu^{l_2-j}\rho]) \rtimes \sigma_1. \end{cases}$$

Now, the first formula in (4.11) implies $i = l_2 - a_-$, or otherwise δ would contain terms $\nu^{-a_-}\rho$ in its supercuspidal support and this would contradict strong positivity of σ. Since $i \geq 0$, we obtain $a_- \leq l_2$.

We consider first the case $l_2 = a_-$. Hence $i = 0$. Since $i \geq j \geq 0$, we obtain also $j = 0$. Now, [Ze] implies that $\delta'' \cong \delta([\nu^{l_1+1}\rho, \nu^a\rho])$. Since $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$, our technical Lemma 4.1 shows that is not possible.

Now, we consider the case $a_- < l_2$. Hence $a_- < a \leq l_2$. Note that $a \geq l_1$, since the first formula in (4.11) results in $\nu^{l_1}\rho \in [\nu^{-a_-}\rho, \nu^a\rho]$.

If $a < l_2$, then we must have $j = 0$. Thus $\delta'' \cong \delta([\nu^{l_1+1}\rho, \nu^a\rho])$. If $a = l_1$, then δ'' is trivial and $\sigma = \sigma''$. Hence the second formula in (4.11) reads $\pi' \leq \delta([\nu^{a_-+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. This contradicts Proposition 3.1 (i), since $a_- < l_1$ and $\mathrm{Jord}_\rho \cap]2l_1+1,\ 2l_2+1[\neq \emptyset$. Thus, we have $l_1 \leq a_- < a < l_2$. We show $a_- = l_1$. If not, then the first formula in (4.11) implies $\delta \cong \delta([\nu^{l_1+1}\rho, \nu^a\rho])$. Hence $\sigma'' \cong \sigma_1$ is a positive discrete series described at the beginning of this section. Also, a is minimal such that $2a+1 \in \mathrm{Jord}_\rho$ and $a > l_1$. This shows $a_- = l_1$, and $a = b_0$. Further, the second formula in (4.11) reads $\pi' \leq \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$. In particular, $a_0 = b_0$ and $\mathrm{Jord}_\rho \cap]2l_1+1,\ 2l_2+1[= \{2a_0+1\}$. Also, Proposition 3.1 (i) applied twice implies

$$\begin{aligned}\pi' &\hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1\\ &\hookrightarrow \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1.\end{aligned} \tag{4.12}$$

Since also

$$\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1,$$

(4.12), using the classification theorem for strongly positive discrete series (see Theorem 1.1), yields

$$\pi' \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

Hence $\pi' \cong \sigma_0$.

If $a = l_2$, then we must have $j > 0$. Now, the first formula in (4.12) shows $l_2 - j \geq l_1$. Since $i \geq j$, we have $l_2 - j \geq l_2 - i = a_- > l_1$. Hence $\delta'' \cong \delta([\nu^{l_1+1}\rho, \nu^{l_2-j}\rho])$. This implies $2(l_2-j)+1 \in \mathrm{Jord}_\rho$, and since $l_2 - j \geq a_-$, we obtain $l_2 - j = a_-$. Lemma 4.1 shows that a_- is minimal such that $2a_-+1 \in \mathrm{Jord}_\rho$ and $a_- > l_1$. In particular, $\mathrm{Jord}_\rho \cap]2l_1+1,\ 2l_2+1[= \{2a_-+1\}$, and $a_- = a_0 = b_0$ Finally, the second formula in (4.12) reads $\pi' \cong \sigma_1$ using Lemma 4.1.

The above discussion shows that only possible tempered irreducible subquotients appear in $\delta \rtimes \sigma$ if and only if $a_0 = b_0$, and only possible tempered subquotients are σ^0 and σ^1. Moreover, they appear with multiplicity at most one since both $\delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \otimes \sigma_0$ and $\delta([\nu^{-a_0}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$ appear in $\mu^*(\delta \rtimes \sigma)$ each with multiplicity one. Finally, Lemma 4.9 show that both σ^0 and σ^1 are actually subquotients of $\delta \rtimes \sigma$. ∎

LEMMA 4.9: *Assume that* $2l_1+1, 2l_2+1 \notin \mathrm{Jord}_\rho$. *If* $a_0 = b_0$, *then* $\delta \rtimes \sigma \geq \sigma^0, \sigma^1$.

Proof: We have the following equivariant morphisms:

$$\begin{aligned}(4.13)\quad & \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1 \\ & \quad \to \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1 \\ & \quad\quad \to \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1.\end{aligned}$$

The first equivariant morphism has the kernel (by the already proved Theorem 4.1 (ii) for σ_1)

$$(4.14)\qquad \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_{temp},$$

where σ_{temp} is the unique common irreducible subquotient of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$ and $\delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma_0$.

The definition of σ^0 and Theorem 1.1 (iii) (see (1.2)) enable us to assume that

$$\sigma^0 \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_{temp},$$

as a unique irreducible subrepresentation.

Next, we observe that

$$\begin{aligned}(4.15)\quad & \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1 \\ & \qquad \cong \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1.\end{aligned}$$

The induced representation on the right-hand side is standard representation and we write L for its Langlands quotient.

If we show that $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ intersects non-trivially the kernel (4.14), then we obtain

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \geq \sigma^0.$$

If not, (4.13) implies

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1.$$

The second equivariant map in (4.13) has the kernel

$$\delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1.$$

The induced representation $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is not contained in that kernel or otherwise

$$Lang(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma) \leq \delta([\nu^{-l_2}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1$$

and this contradicts Theorem 2.1 (see the definition of σ_1 and the assumption of Lemma 4.9).

Now, applying the second equivariant map in (4.13) we obtain non-trivial equivariant map

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \to \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1.$$

Since, by Proposition 3.1,

$$\delta([\nu^{-a_0}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1 \cong \delta([\nu^{l_1+1}\rho, \nu^{a_0}\rho]) \rtimes \sigma_1,$$

we obtain a non-trivial equivariant map

$$\begin{aligned}\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \to \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-a_0}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1 \\ \cong \delta([\nu^{-a_0}\rho, \nu^{-l_1-1}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma_1.\end{aligned}$$

The induced representation on the right-hand side has L defined after (4.15) as a unique Langlands subrepresentation. In particular,

$$L \leq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

and this contradicts Lemma 4.7.

Now, we prove $\delta \rtimes \sigma \geq \sigma^1$. First, we observe the following equivariant morphism:

(4.16)

$$\delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma \to \delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

The first induced representation in (4.16) has the following subrepresentation:

$$\delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{temp}, \tag{4.17}$$

where σ_{temp} is the unique common irreducible subquotient of $\delta([\nu^{-a_0}\rho, \nu^{l_1}\rho]) \rtimes \sigma$ and $\delta([\nu^{-a_0}\rho, \nu^{a_0}\rho]) \rtimes \sigma_0$. Note that σ^1 can be taken to be a subrepresentation of (4.17).

The equivariant morphism in (4.16) has the kernel

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

We show that σ^1 is a subrepresentation of that kernel. If not, we obtain

$$\sigma^1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \times \delta([\nu^{a_0+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.$$

This implies that for some irreducible representation σ'',

$$\sigma^1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{a_0}\rho]) \rtimes \sigma''.$$

This implies ([Mœ], Lemma 3.1)

$$\epsilon^1(2a_0+1,\rho)\cdot\epsilon^1(2l_1+1,\rho)^{-1}=1,$$

which contradicts the definition of σ^1. ∎

5. Strongly positive discrete series III

Let $\delta \rtimes \sigma$ be the induced representation given by (2.1)–(2.3) and assume that σ is a strongly positive discrete series. In this section we determine the structure of $\delta \rtimes \sigma$ assuming

$$l_1 = -1/2.$$

It is more convenient to write $l = l_2$. Thus, in new notation we determine the structure of

$$\delta([\nu^{1/2}\rho,\ \nu^{l}\rho]) \rtimes \sigma,$$

where $l \in 1/2 + \mathbb{Z}_{\geq 0}$, σ is a strongly positive discrete series, where $\mathrm{Jord}_\rho \neq \emptyset$ and $l - l' \in \mathbb{Z}$, for any $2l'+1 \in \mathrm{Jord}_\rho$ (see (2.1)–(2.3)).

We define b_0 by $2b_0+1 = \min \mathrm{Jord}_\rho$. As in the previous section we define five discrete series.

If $\epsilon(2b_0+1,\rho)=1$, we define a strongly positive discrete series σ_1 by

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_1) = \mathrm{Jord}_{\rho'}, & \rho' \neq \rho, \\ \mathrm{Jord}_{\rho}(\sigma_1) = \mathrm{Jord}_{\rho}\setminus\{2b_0+1\}. \end{cases} \tag{5.1}$$

It follows immediately from the definition of an alternated triple (see Section 1) that σ_1 is well-defined with (5.1). Moreover, $\epsilon_{\sigma_1}(\min \mathrm{Jord}_\rho(\sigma_1),\rho) = -1$, and

$$\sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1 \tag{5.2}$$

(see Theorem 1.1 (ii)).

Assume that $\mathrm{Jord}_\rho \cap [2, 2l+1] = \emptyset$ or $\mathrm{Jord}_\rho \cap [2,2l+1] = \{2b_0+1\}$, $l \neq b_0$. We also define a discrete series σ_2 specifying its admissible triple $(\mathrm{Jord}(\sigma_2), \sigma', \epsilon_{\sigma_2})$. First,

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord} \cup \{(2l+1,\rho)\},$$

and ϵ_{σ_2} extends ϵ_{σ_1} such that

$$\epsilon_{\sigma_2}(2l+1,\rho)\cdot\epsilon_{\sigma_2}(2b_0+1,\rho)^{-1} = 1,$$

and all other products $\epsilon_{\sigma_2}(b,\rho)\cdot\epsilon_{\sigma_2}(b_-,\rho)^{-1}=-1$, $(b,\rho)\in \mathrm{Jord}(\sigma_2)$. In fact, we have

$$\begin{cases} \sigma_2 \hookrightarrow \delta([\nu^{-l}\rho,\nu^{b_0}\rho])\rtimes\sigma_1, & b_0>l,\\ \sigma_2 \hookrightarrow \delta([\nu^{-b_0}\rho,\nu^{l}\rho])\rtimes\sigma_1, & l>b_0. \end{cases}$$

If $\epsilon(2b_0+1,\rho)=-1$ and $l\neq b_0$, we define a strongly positive discrete series σ_0 by

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_0)=\mathrm{Jord}_{\rho'}, & \rho'\neq\rho,\\ \mathrm{Jord}_{\rho}(\sigma_0)=\mathrm{Jord}_{\rho}\cup\{2l+1\}. \end{cases} \tag{5.3}$$

It follows immediately from the definition of an alternated triple (see Section 1) that σ_0 is well-defined with (5.3). Obviously, $\epsilon_{\sigma_0}(\min \mathrm{Jord}_\rho(\sigma_0),\rho)=1$, and if $l<b_0$, then

$$\sigma_0\hookrightarrow\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma. \tag{5.4}$$

Assume that

$$\mathrm{Jord}_\rho\cap[2,\ 2l+1[\ \neq\emptyset,$$

and write $2a_0+1$ for the largest element in that intersection. If $2l+1\notin\mathrm{Jord}_\rho$, then we also denote by σ_3 strongly positive discrete series defined by (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_3)=\mathrm{Jord}_{\rho'}(\sigma), & \rho'\neq\rho,\\ \mathrm{Jord}_{\rho}(\sigma_3)=\mathrm{Jord}_{\rho}(\sigma)\setminus\{2a_0+1\}\cup\{2l+1\}. \end{cases}$$

Under the same assumptions, we define strongly positive discrete series σ_4 by (see Proposition 1.2)

$$\begin{cases} \mathrm{Jord}_{\rho'}(\sigma_4)=\mathrm{Jord}_{\rho'}(\sigma_1), & \rho'\neq\rho,\\ \mathrm{Jord}_{\rho}(\sigma_4)=\mathrm{Jord}_{\rho}(\sigma_1)\setminus\{2a_0+1\}\cup\{2l+1\}. \end{cases}$$

The main result of this section is the following theorem.

THEOREM 5.1: *Under the above assumptions we have the following.*

(i) *Assume $\epsilon(2b_0+1,\rho)=-1$. Then we have in the appropriate Grothendieck group*

$$\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma=$$
$$\begin{cases} Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+\sigma_0, & l<b_0,\\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma), & l=b_0,\\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+Lang(\delta([\nu^{1/2}\rho,\nu^{a_0}\rho])\rtimes\sigma_3), & \\ \qquad\qquad l>b_0,\ 2l+1\notin\mathrm{Jord}_\rho, & \\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma), & l>b_0,\ 2l+1\in\mathrm{Jord}_\rho. \end{cases}$$

(ii) *Assume* $\epsilon(2b_0+1,\rho)=1$. *If* $l=b_0$, *then*

$$\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma \quad \text{and} \quad \delta([\nu^{-b_0}\rho,\nu^{b_0}\rho])\rtimes\sigma_1$$

have a unique common irreducible subquotient. Let us denote it by σ_{temp}. *Moreover, in the appropriate Grothendieck group, we have*

$$\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma=$$
$$\begin{cases} Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+\sigma_2, & l<b_0,\\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+\sigma_{temp}, & l=b_0,\\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+Lang(\delta([\nu^{-b_0}\rho,\nu^{l}\rho])\rtimes\sigma_1), & \\ \qquad\qquad l>b_0, 2l+1\in \mathrm{Jord}_\rho, & \\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+Lang(\delta([\nu^{1/2}\rho,\nu^{a_0}\rho])\rtimes\sigma_3) & \\ +Lang(\delta([\nu^{-b_0}\rho,\nu^{l}\rho])\rtimes\sigma_1)+\sigma_2, \quad l>b_0, a_0=b_0, 2l+1\notin\mathrm{Jord}_\rho, & \\ Lang(\delta([\nu^{1/2}\rho,\nu^{l}\rho])\rtimes\sigma)+Lang(\delta([\nu^{1/2}\rho,\nu^{a_0}\rho])\rtimes\sigma_3) & \\ +Lang(\delta([\nu^{-b_0}\rho,\nu^{l}\rho])\rtimes\sigma_1)+Lang(\delta([\nu^{-b_0}\rho,\nu^{a_0}\rho])\rtimes\sigma_4), & \\ \qquad\qquad l>b_0, a_0>b_0, 2l+1\notin\mathrm{Jord}_\rho. & \end{cases}$$

We prove the theorem in the next few lemmas.

LEMMA 5.1: *Assume* $\epsilon(2b_0+1,\rho)=-1$. *Then* $\delta([\nu^{1/2}\rho,\ \nu^{l}\rho])\rtimes\sigma$ *has a tempered irreducible subquotient if and only if* $l<b_0$. *That tempered subquotient is a strongly positive discrete series* σ_0 *defined by (5.3). It appears with multiplicity one* (see [MT], Lemma 4.1).

Proof: If $l=b_0$, then any possible tempered irreducible subquotient must be tempered (arguing as in [MT], Section 8), and it would have a discrete series part with not enough Jordan blocks as required in the definition of an admissible triple (see Section 1 and Lemma 2.1).

If $l\neq b_0$, only a possible tempered irreducible subquotient is a strongly positive discrete series σ_0 defined above. We show that σ_0 is a subquotient of $\delta([\nu^{1/2}\rho,\ \nu^{l}\rho])\rtimes\sigma$ if and only if $l<b_0$.

Assume that σ_0 is a subquotient of $\delta([\nu^{1/2}\rho,\ \nu^{l}\rho])\rtimes\sigma$ when $l>b_0$. First, since by definition $\epsilon_{\sigma_0}(\min\mathrm{Jord}_\rho(\sigma_0),\rho)=1$, we must have

$$\sigma_0\hookrightarrow\delta([\nu^{1/2}\rho,\nu^{b_0}\rho])\rtimes\sigma_1, \tag{5.5}$$

where σ_{10} is a strongly positive discrete series defined by

$$\begin{cases}\mathrm{Jord}_{\rho'}(\sigma_{10})=\mathrm{Jord}_{\rho'}(\sigma_0), & \rho'\neq\rho,\\ \mathrm{Jord}_{\rho}(\sigma_{10})=\mathrm{Jord}_{\rho}(\sigma_0)\setminus\{2b_0+1\}.\end{cases}$$

Since $\sigma_0 \leq \delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma$, we see that Frobenius reciprocity and (5.5) implies

$$\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \otimes \sigma_{10} \leq \mu^*(\delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma).$$

We analyze this using Theorem 1.2. So, let $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$ be an irreducible representation, and indices $0 \leq j \leq i \leq l + 1/2$. We have the following formulae:

$$\begin{cases} \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \leq \delta([\nu^{i-l}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{l-j}\rho, \nu^l\rho]) \times \delta'', \\ \sigma_{10} \leq \delta([\nu^{l+1-i}\rho, \nu^{l-j}\rho]) \rtimes \sigma''. \end{cases}$$

Since the left-hand side in the first displayed formula does not have negative terms $i = l + 1/2$, and since $l > b_0$, we must have $j = 0$. This implies $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^{b_0}\rho])$. This is a contradiction, since $\epsilon(2b_0 + 1, \rho) = -1$ implies that σ does not have terms $\nu^{\pm 1/2}\rho$ in its supercuspidal support. ∎

COROLLARY 5.1: *Assume* $\epsilon(2b_0 + 1, \rho) = -1$. *Then* $\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma$ *is irreducible.*

Proof: Lemma 5.1 shows that it has no tempered subquotient. Lemma 2.2 can be easily applied to show that its only non-tempered irreducible subquotient is its Langlands quotient. ∎

The next lemma is an analogue of Lemma 4.1.

LEMMA 5.2: *Assume that* $\mu^*(\sigma) \geq \delta([\nu^{1/2}\rho, \nu^l\rho]) \otimes \sigma''$, *where* $l \in 1/2 + \mathbb{Z}_{\geq 0}$, *and* σ'' *is irreducible. Then* $\epsilon(2b_0 + 1, \rho) = 1$, $l = b_0$, *and* $\sigma'' \cong \sigma_1$.

Proof: Since $\epsilon(2b_0 + 1, \rho) = -1$ implies that σ does not have terms $\nu^{\pm 1/2}\rho$ in its supercuspidal support, we see that $\epsilon(2b_0 + 1, \rho) = -1$. First, $\mu^*(\sigma) \geq \delta([\nu^{1/2}\rho, \nu^l\rho]) \otimes \sigma''$ implies that there exists an irreducible representation ${\sigma''}_1$ such that

$$\sigma \hookrightarrow \nu^l\rho \times \cdots \times \nu^{1/2}\rho \rtimes {\sigma''}_1. \tag{5.6}$$

First (5.6) and ([Mœ], Remark 5.1.2) implies $2l + 1 \in \mathrm{Jord}_\rho$, and hence $l \geq b_0$. Since σ is strongly positive, ${\sigma''}_1$ is also strongly positive (see Proposition 1.1). Arguing as in ([MT], Section 8), we see that

$$\begin{cases} \mathrm{Jord}_{\rho'}({\sigma''}_1) = \mathrm{Jord}_{\rho'}, \quad \rho' \not\cong \rho, \\ \mathrm{Jord}_\rho({\sigma''}_1) = \mathrm{Jord}_\rho \setminus \{2l+1\}. \end{cases} \tag{5.7}$$

Since ${\sigma''}_1$ is strongly positive it is completely determined by (5.7) (see Proposition 1.2) and $\epsilon_{{\sigma''}_1}(\min \mathrm{Jord}_\rho({\sigma''}_1), \rho) = -1$. Now, using Corollary 5.1, we can show that in fact

$$\sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes {\sigma''}_1 \tag{5.8}$$

as in the proof of the corresponding fact in Lemma 4.1. We leave the details to the reader.

We now show $l = b_0$. If not, (5.8) and Corollary 5.1 implies the following:

$$\begin{aligned}\sigma &\hookrightarrow \delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma''_1\\ &\hookrightarrow \delta([\nu^{b_0+1}\rho, \nu^l\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma''_1\\ &\cong \delta([\nu^{b_0+1}\rho, \nu^l\rho]) \times \delta([\nu^{-b_0}\rho, \nu^{-1/2}\rho]) \rtimes \sigma''_1\\ &\cong \delta([\nu^{-b_0}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{b_0+1}\rho, \nu^l\rho]) \rtimes \sigma''_1.\end{aligned}$$

This implies $\sigma \hookrightarrow \nu^{-1/2}\rho \rtimes \sigma'''$, for some irreducible representation σ'''. This contradicts the square integrable criterion. Finally, $\sigma'' \cong \sigma''_1$ follows from ([M3], Theorem 2.3). ∎

We write $\delta = \delta([\nu^{1/2}\rho,\ \nu^l\rho])$ as before. The next two lemmas determine all tempered irreducible subquotients of $\delta \rtimes \sigma$, if $\epsilon(2b_0 + 1, \rho) = 1$.

LEMMA 5.3: *Assume that $2l + 1 \notin \mathrm{Jord}_\rho$. Then $\delta \rtimes \sigma$ contains a tempered irreducible subquotient if and only if $\mathrm{Jord}_\rho \cap [2, 2l + 1] = \emptyset$ or $\mathrm{Jord}_\rho \cap [2, 2l + 1] = \{2b_0 + 1\}$. That irreducible subquotient is a discrete series σ_2 defined at the beginning of this section. Moreover, σ_2 appears in the composition series of $\delta \rtimes \sigma$ exactly once.*

Proof: First, Lemma 2.1 shows that tempered irreducible subquotients must be in discrete series. Their Jordan blocks must be $\mathrm{Jord}(\sigma) \cup \{(2l + 1, \rho)\}$.

Let π be one of them. To determine π, we need to compute its partial ϵ_π function. First, π cannot be strongly positive and therefore there must be $(2a + 1, \rho) \in Jord(\pi)$ such that $(2a + 1)_- := 2a_- + 1$ is defined, and

$$\pi \hookrightarrow \delta([\nu^{-a_-}\rho, \nu^a\rho]) \rtimes \pi',$$

where π' is a discrete series such that

$$\mathrm{Jord}(\pi') = \mathrm{Jord}(\pi) \setminus \{(2a + 1, \rho), (2a_- + 1, \rho)\},$$

and its $\epsilon_{\pi'}$ is the restriction of ϵ_π of π to $\mathrm{Jord}(\pi')$. Now, Frobenius reciprocity implies that

$$\delta([\nu^{-a_-}\rho, \nu^a\rho]) \otimes \pi' \leq \mu^*(\delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma).$$

We analyze this using Theorem 1.2. So, let $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$ be an irreducible representation, and indices $0 \leq j \leq i \leq l + 1/2$. We have the following formulae:

$$(5.9)\qquad \begin{cases}\delta([\nu^{-a_-}\rho, \nu^a\rho]) \leq \delta([\nu^{i-l}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{l+1-j}\rho, \nu^l\rho]) \times \delta'',\\ \pi' \leq \delta([\nu^{l+1-i}\rho, \nu^{l-j}\rho]) \rtimes \sigma''.\end{cases}$$

Now, the first formula in (5.9) implies $i = l - a_-$, or otherwise δ would contain terms $\nu^{-a_-}\rho$ in its supercuspidal support and this would contradict strong positivity of σ (see Proposition 1.1). (Note that $l+1-j \geq 1/2$.) Since $i \geq 0$, we obtain $a_- \leq l$.

We consider first the case $a_- = l$. Hence $i = 0$. Since $i \geq j \geq 0$. We obtain also $j = 0$. Now, [Ze] implies that $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^a\rho])$. Since $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$, Lemma 5.2 shows $a = b_0$ and $\sigma'' \cong \sigma_1$.

Now, we consider the case $a_- < l$. Hence $a_- < a \leq l$.

If $a < l$, then we must have $j = 0$. Thus $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^a\rho])$. Hence $a = b_0$ by Lemma 5.2. This implies $a_- < b_0$. This is not possible.

If $a = l$, then we must have $j > 0$. Note that $l - j \geq l - i = a_- \geq b_0 \geq 1/2$. Hence $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^{l-j}\rho])$. Lemma 5.1 implies $b_0 = l - j$. This implies $a_- = b_0$ and $i = j = l - a_0$.

The above discussion shows that necessary conditions for the existence of a tempered irreducible subquotient must hold. It also shows that the only possible irreducible subquotient is a discrete series σ_2 defined at the beginning of this section. Moreover, σ_2 appears in the composition series of $\delta \rtimes \sigma$ at most once. We need to show that it actually appears. We do that only in the case $l < b_0$. The other case goes the same way. First, we observe the following:

$$(5.10) \qquad \begin{cases} \delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma \leq \delta([\nu^{1/2}\rho, \nu^l\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1, \\ \delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1 \leq \delta([\nu^{1/2}\rho, \nu^l\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1. \end{cases}$$

The first formula in (5.10) follows from (5.2). The second formula follows using [Ze] and usual properties of induced representations of classical groups.

We show that $\delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ appears in

$$(5.11) \qquad \mu^*(\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1)$$

with multiplicity exactly two. Also, the classification of discrete series [MT] shows that the appropriate Jacquet module of $\delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$ contains $\delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ at least twice. Now, combining with (5.10) we see that $\delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$ and $\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma$ must have a unique common be an irreducible subquotient.

Let us now compute the multiplicity of $\delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ in (5.11). We use Theorem 2.1. So, let $\mu^*(\sigma_1) \geq \delta'' \otimes \sigma''$ be an irreducible representation, and indices $0 \leq j \leq i \leq l + 1/2$, $0 \leq j' \leq i' \leq b_0 + 1/2$. We have the following formulae:

$$(5.12) \quad \begin{cases} \delta([\nu^{-l}\rho, \nu^{b_0}\rho]) \leq \delta([\nu^{i-l}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{l+1-j}\rho, \nu^l\rho]) \times \delta([\nu^{i'-b_0}\rho, \nu^{-1/2}\rho]) \\ \qquad \times \delta([\nu^{b_0+1-j'}\rho, \nu^{b_0}\rho]) \times \delta'' \end{cases}$$

and

$$\sigma_1 \le \delta([\nu^{l+1-i}\rho, \nu^{l-j}\rho]) \times \delta([\nu^{b_0+1-i}\rho, \nu^{b_0-j}\rho]) \rtimes \sigma''. \tag{5.13}$$

Now, the first formula in (5.12) implies $i = 0$ or $i' = b_0 - l$, or otherwise δ would contain terms $\nu^{-b_0}\rho$ in its supercuspidal support and this would contradict strong positivity of σ_1 (see Proposition 1.1). (Note that $l+1-j \ge 1/2$, $b_0+1-j \ge 1/2$.)

Assume first $i = 0$. Then since $i \ge j \ge 0$ we obtain also $j = 0$. We also see from (5.12) that $i' = b_0 + 1$, or otherwise the left-hand side of (5.12) would contain $\nu^{-1/2}\rho$ in its supercuspidal support twice. Finally, $b_0+1-j' = 1/2$, that is $j' = b_0+1/2$ or $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^{b_0-j'}\rho])$, and this implies $2(b_0-j')+1 \in \mathrm{Jord}_\rho(\sigma_1)$ ([Mœ], Remark 5.1.2), a contradiction. Thus, δ'' is trivial and $\sigma'' \cong \sigma_1$. Obviously (5.13) holds. This produces $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ once.

Consider the case $i' = b_0 - l$. This implies $i = l + 1/2$, or otherwise the left-hand side of (5.12) would contain $\nu^{-1/2}\rho$ in its supercuspidal support twice. If $j = 0$, then we must have $j' = b_0 + 1/2$ or $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^{b_0-j'}\rho])$, and this implies $2(b_0-j')+1 \in \mathrm{Jord}_\rho$ ([Mœ], Remark 5.1.2), a contradiction. Thus $j > 0$. We see that $j = l + 1/2$ or $2(b_0 - j') + 1 \in \mathrm{Jord}_\rho(\sigma_1)$, a contradiction. Also, (5.12) implies that $j' > 0$ or $2b_0 + 1 \in \mathrm{Jord}_\rho(\sigma_1)$, and this is a contradiction. Now, (5.12) implies that $l = b_0 - j$. Obviously (5.13) holds. This produces $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ one more time. ∎

LEMMA 5.4: *Assume that $2l + 1 \in \mathrm{Jord}_\rho$. Then $\delta \rtimes \sigma$ contains a tempered irreducible subquotient if and only if $l = b_0$ (that is, $2l + 1 = \min \mathrm{Jord}_\rho$). That tempered subquotient is a common irreducible subquotient of $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$, where σ_1 is defined by (5.1), and $\delta \rtimes \sigma$. It appears in the composition series of $\delta \rtimes \sigma$ exactly once.*

Proof: We first analyze possible tempered irreducible subquotients. Let π be one of them. The assumption of the lemma implies that

$$\pi \hookrightarrow \delta([\nu^{-a-}\rho, \nu^{a}\rho]) \rtimes \pi',$$

where π' is a discrete series. Now, Frobenius reciprocity implies

$$\delta([\nu^{-l}\rho, \nu^{l}\rho]) \otimes \pi' \le \mu^*(\delta([\nu^{1/2}\rho, \nu^{l}\rho]) \rtimes \sigma).$$

We analyze this using Theorem 1.2. So, let $\mu^*(\sigma) \ge \delta'' \otimes \sigma''$ be an irreducible representation, and indices $0 \le j \le i \le l + 1/2$. We have the following formulae:

$$\begin{cases} \delta([\nu^{-l}\rho, \nu^{l}\rho]) \le \delta([\nu^{i-l}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{l+1-j}\rho, \nu^{l}\rho]) \times \delta'', \\ \pi' \le \delta([\nu^{l+1-i}\rho, \nu^{l-j}\rho]) \rtimes \sigma''. \end{cases} \tag{5.14}$$

Now, the first formula in (5.14) implies $i = 0$, or otherwise δ would contain terms $\nu^{-l}\rho$ in its supercuspidal support and this would contradict strong positivity of σ (see Proposition 1.1). (Note that $l+1-j \geq 1/2$.) Since $i \geq j \geq 0$, we obtain also $j = 0$. Now, [Ze] implies that $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^l\rho])$, since $\mu^*(\sigma) \geq \delta'' \otimes \sigma''$. Lemma 5.2 shows $l = b_0$ and $\sigma'' \cong \sigma_1$. The second formula in (5.14) implies $\pi' \cong \sigma''$. Again, using Lemma 5.1, we obtain $\pi' \cong \sigma_1$ defined in (5.1). The above computation also shows that π is a common irreducible subquotient of $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$ and $\delta \rtimes \sigma$. Furthermore, since $\delta([\nu^{-l}\rho, \nu^l\rho]) \otimes \sigma_1$ is contained in $\mu^*(\delta([\nu^{1/2}\rho, \nu^l\rho]) \rtimes \sigma)$ with multiplicity one, the tempered irreducible subquotient must appear in the composition series of $\delta \rtimes \sigma$ exactly once.

It remains to show that it actually appears. To finish, we observe the following:

$$\text{(5.15)} \qquad \begin{cases} \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma \leq \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1, \\ \delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1 \leq \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1. \end{cases}$$

The first formula in (5.15) follows from (5.2). The second formula follows using [Ze] and usual properties of induced representation of classical groups.

We show that $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ appears in

$$\text{(5.16)} \qquad \mu^*(\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \times \delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1)$$

with multiplicity exactly two. Then since $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$ reduces (since $2b_0 + 1 \notin \mathrm{Jord}_\rho(\sigma_1)$), Frobenius reciprocity implies that $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ appears in its appropriate Jacquet module twice. Now, combining with (5.15) we see that $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \rtimes \sigma_1$ and $\delta([\nu^{1/2}\rho, \nu^{b_0}\rho]) \rtimes \sigma$ must have a unique common irreducible subquotient.

Let us now compute the multiplicity of $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ in (5.16). We use Theorem 1.2. So, let $\mu^*(\sigma_1) \geq \delta'' \otimes \sigma''$ be an irreducible representation, and indices $0 \leq j \leq i \leq b_0 + 1/2$, $0 \leq j' \leq i' \leq b_0 + 1/2$. We have the following formulae:

$$\text{(5.17)} \qquad \begin{cases} \delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \leq \delta([\nu^{i-b_0}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{b_0+1-j}\rho, \nu^{b_0}\rho]) \\ \qquad \times \delta([\nu^{i'-b_0}\rho, \nu^{-1/2}\rho]) \times \delta([\nu^{b_0+1-j'}\rho, \nu^{b_0}\rho]) \times \delta'' \end{cases}$$

and

$$\text{(5.18)} \qquad \sigma_1 \leq \delta([\nu^{b_0+1-i}\rho, \nu^{b_0-j}\rho]) \times \delta([\nu^{b_0+1-i'}\rho, \nu^{b_0-j'}\rho]) \rtimes \sigma''.$$

Now, the first formula in (5.17) implies $i = 0$ or $i' = 0$, or otherwise δ'' would contain terms $\nu^{-b_0}\rho$ in its supercuspidal support and this would contradict strong positivity of σ_1 (see Proposition 1.1). (Note that $b_0+1-j \geq 1/2$, $b_0+1-j' \geq 1/2$.)

Assume first $i = 0$. Then since $i \geq j \geq 0$ we obtain also $j = 0$. We also see from (5.17) that $i' = b_0 + 1$, or otherwise the left-hand side of (5.17) would contain $\nu^{-1/2}\rho$ in its supercuspidal support twice. Finally, $b_0 + 1 - j' = 1/2$, that is $j' = b_0 + 1/2$ or $\delta'' \cong \delta([\nu^{1/2}\rho, \nu^{b_0 - j'}\rho])$, and this implies $2(b_0 - j') + 1 \in \mathrm{Jord}_\rho$ ([Mœ], Remark 5.1.2), a contradiction. Thus, δ'' is trivial and $\sigma'' \cong \sigma_1$. Obviously (5.18) holds. This produces $\delta([\nu^{-b_0}\rho, \nu^{b_0}\rho]) \otimes \sigma_1$ once.

The case $i' = 0$ goes the same way. ∎

In the next lemma we determine the non-tempered irreducible subquotients of $\delta([\nu^{1/2}\rho, \nu^{l}\rho]) \rtimes \sigma$. First, Lemma 2.2 shows that the only non-tempered irreducible quotient is its Langlands quotient unless $\mathrm{Jord}_\rho \cap [2,\ 2l+1[\ \neq \emptyset$. We assume that

$$\mathrm{Jord}_\rho \cap [2,\ 2l+1[\ \neq \emptyset.$$

We omit the proofs completely, since they are analogous to that of similar results in Section 4.

LEMMA 5.5: *Assume that* $\mathrm{Jord}_\rho \cap [2,\ 2l+1[\ \neq \emptyset$. *(Hence $b_0 < l$.) Then all non-tempered irreducible subquotients of* $\delta([\nu^{1/2}\rho, \nu^{l}\rho]) \rtimes \sigma$ *come exactly once in its composition series, and they are given by:*

$$\begin{cases} Lang(\delta([\nu^{1/2}\rho, \nu^{l}\rho]) \rtimes \sigma), & \\ Lang(\delta([\nu^{1/2}\rho, \nu^{a_0}\rho]) \rtimes \sigma_3), & \text{if } 2l+1 \notin \mathrm{Jord}_\rho, \\ Lang(\delta([\nu^{-b_0}\rho, \nu^{l}\rho]) \rtimes \sigma_1), & \text{if } \epsilon(2b_0+1,\rho) = 1, \\ Lang(\delta([\nu^{-b_0}\rho, \nu^{a_0}\rho]) \rtimes \sigma_4), & \text{if } \epsilon(2b_0+1,\rho) = 1,\ b_0 < a_0, 2l+1 \notin \mathrm{Jord}_\rho. \end{cases}$$

References

[Ca] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint.

[J] C. Jantzen, *Degenerate principal series for symplectic and odd-orthogonal groups*, Memoirs of the American Mathematical Society **590** (1996), 1–100.

[M1] G. Muić, *Some results on square integrable representations; Irreducibility of standard representations*, International Mathematics Research Notices **14** (1998), 705–726.

[M2] G. Muić, *A proof of Casselman–Shahidi's conjecture for quasi-split classical groups*, Canadian Mathematical Bulletin **44** (2001), 298–312.

[M3] G. Muić, *Howe correspondence for discrete series representations; the case of* $(Sp(n), O(V))$, Journal für die reine und angewandte Mathematik, to appear.

[M4] G. Muić, *Composition series of generalized principal series; The general case*, in preparation.

[M5] G. Muić, *Reducibility of generalized principal series*, Canadian Journal of Mathematics, to appear.

[Mœ] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, Journal of the European Mathematical Society **4** (2002), 143–200.

[MT] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups*, Journal of the American Mathematical Society **15** (2002), 715–786.

[Sh] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Annals of Mathematics **132** (1990), 273–330.

[Sh1] F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Mathematical Journal **66** (1992), 1–41.

[T] M. Tadić, *On reducibility of parabolic induction*, Israel Journal of Mathematics **107** (1998), 29–91.

[W] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques, d'après Harish-Chandra*, Journal of the Institute of Mathematics of Jussieu **2** (2003), 235–333.

[Ze] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. On irreducible representations of $GL(n)$*, Annales Scientifiques de l'École Normale Supérieure **13** (1980), 165–210.